Differential Equations and Vector Calculus (MTH 312): Notes

Adam Szava

Fall 2021

Introduction

This is my compilation of notes from Differential Equations and Vector Calculus (MTH 312) from Ryerson University. All information comes from my professor's lectures, the textbooks *Elementary Differential Equations with Boundary Value Problems*, *OpenStax: Calculus Volume 3*, and online resources.

In this text, definitions are boxed in red, theorems are boxed in blue, and any examples are boxed in green.

Introductory Topics and Definitions

The following section is a collection of definitions and discussions which I feel need to be properly defined before you can being this course, however they were not included in the actual course material. All information comes from the textbook *Advanced Engineering Mathematics by Dennis G. Zill*. I also recommend this textbook to be used for further reading.

We begin with the most important definition of the course, the formal definition of a differential equation:

Definition 0.1.1 Differential Equations

An equation containing derivatives of one or more dependent variables with respect to one or more independent variables is a differential equation (DE).
A DE with only one independent variable is called an **ordinary** DE (ODE).
A DE with multiple independent variables is called a **partial** DE (PDE).
The order of a DE is the order of the highest order derivative in the equation.

There are a few forms you can write a differential equation.

1. **General Form** for n^{th} order ODEs:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

... where F is a function of $n+2$ variables. We assume in this course that it is always possible to rearrange general form into **normal form**...

2. **Normal Form** for n^{th} order ODEs:

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$$

3. **Differential Form** for *first* order ODEs:

$$M(x, y)dx + N(x, y)dy = 0$$

These two equations are the key to solving DEs:

$$\frac{dy}{dx} = y'$$

$$dy = y'dx$$

... this may seem obvious if you think of the differential as a fraction, which is fine to do for our purposes.

You can convert between *differential form* and *normal form* by using the previous equations:

$$M(x, y)dx + N(x, y)dy = 0$$

$$N(x, y)dy = -M(x, y)dx$$

$$dy = -\frac{M(x, y)}{N(x, y)}dx$$

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

... and making the substitution $f(x, y) = -\frac{M(x,y)}{N(x,y)}$:

$$y' = f(x, y)$$

... which is exactly the definition of a first order ODE in normal form.

Definition 0.1.2 Solutions to a DE

Any function ϕ, defined on some interval I possessing n derivatives that are continuous on that interval I, which when substituted into the DE it becomes an identity, is a **solution** to the DE on I.

Any function that can be substituted into the DE and the resulting equation is true, is a solution. You must define what interval the solution is defined on, because it does matter.

Definition 0.1.3 Implicit Solutions

A relation $G(x, y) = 0$ is an **implicit solution** to an ODE on I **if** there exists a function $y = \phi(x)$ that satisfies both the relation, and the ODE on I. In this course we assume that that function $\phi(x)$ always exists.

You can implicitly differentiate the relation $G(x, y) = 0$ to find the derivatives of y with respect to x, and substitute that into the DE.

A DE can have a family of solutions which come from an arbitrary parameter in it's explicit or implicit form. This corresponds to an infinite set of solutions. An **exact** solution is a solution without a parameter. **Singular** solutions do not belong to any families and are usually trivial.

Review of Differential Equation Content from MTH 240

4.3 Separable Differential Equations

This section begins with a lot of definitions to familiarize with terminology relating to differential equations.

Definition 1.1.1 Terminology

A **Differential Equation** (DE) is an equation in variables x, y, y', y'', y''', etc, where y is a function of x.
A **solution to a DE** is a function $f(x)$ where subbing in $y = f(x)$ makes an equation which is always true.
The **order of a DE** is n where the n^{th} derivative of y is the highest order term which appears.
The **general solution to a DE** is a parametrized family of functions.
An **initial value problem** consists of a DE with a specific condition. A solutions to this type of problem is a solution to the DE that satisfies the conditions.

Definition 1.1.2 Seperable D.E.

A DE is **separable** if it can be rearranged into the form:

$$y' = f(x)\ g(y)$$

As in you can *separate* x and y.

In order to solve a separable differential equation, you follow this formula:

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx$$

Do the integration, and rearrange for y if possible.

4.5 Linear Differential Equations

Definition 1.2.1 Linear D.E. (L.D.E.)

A **linear differential equation** (LDE) is a DE which can be written in the form:

$$y' + p(x)y = q(x)$$

Notably the coefficient on the y' term is 1. In order to find a solution to this type of DE, we must first define an Integrating Factor.

Definition 1.2.2 Integrating Factor

In general (not just for LDEs) an *Integrating Factor* is a function usually denoted $I(x)$ which, when multiplied on both sides of a DE, allows both sides to be integrated. For LDEs, the *Integrating Factor* will always be in the form:

$$I(x) = e^{\int p(x)\ dx}$$

Where $p(x)$ is the same $p(x)$ as in the LDE.

This helps us solve the LDE because of the following:

$$y' + p(x)y = q(x)$$

$$e^{\int p(x)\ dx}y' + e^{\int p(x)\ dx}p(x)y = e^{\int p(x)\ dx}q(x)$$

$$\int e^{\int p(x)\ dx} y' + e^{\int p(x)\ dx} p(x) y\ dx = \int e^{\int p(x)\ dx} q(x)\ dx$$

Notice the the left side will always be the result of a product rule differentiation. So:

$$e^{\int p(x)\ dx} y = \int e^{\int p(x)\ dx} q(x)\ dx$$

$$y = \frac{1}{e^{\int p(x)\ dx}} \int e^{\int p(x)\ dx} q(x)\ dx$$

And now we have the solution to the LDE.

Chapter 2: First Order Equations

Differential equations model the real world. Solving a differential equation provides and interpretation of a real phenomenon. The first part of this course is concerned with beginning the study of solving these differential equations.

2.2 Separable Differential Equations

This course begins with another section on separable D.E.

Definition 2.1.1 Seperable DE

A first order DE in the form:

$$y' = f(x, y)$$

Is seperable if and only if $f(x, y)$ can be written as:

$$f(x, y) = h(x)g(y)$$

So the DE can be written in the form:

$$y' = h(x)g(y)$$

In this course we often abuse the differential notation, treating it as a fraction. For our purposes you can.

To solve a separable DE, we look at the general form and solve for $y(x)$:

$$y' = h(x)g(y)$$

$$\frac{dy}{dx} = h(x)g(y)$$

$$\frac{1}{g(y)} dy = h(x)dx$$

$$\int \frac{1}{g(y)} dy = \int h(x) dx$$

$$G(y) = H(x) + c$$

If you can rearrange the following equation for y then do so and you will have an *explicit* formula for $y(x)$. If you cannot, then leave the equation in the form:

$$G(y) - H(x) = c$$

$$P(x, y) = c$$

... which is a function which *implicitly* solves the DE. This means that the function $P(x, y)$ can be *implicitly* differentiated to show that it solves the DE.

2.5 Exact Differential Equations

Given some first order differential equation in the form:

$$y' = \frac{P(x, y)}{Q(x, y)}$$

... we can choose to write it as the following:

$$\frac{dy}{dx} = \frac{P(x, y)}{M(x, y)}$$

$$M(x, y)dy = P(x, y)dx$$

$$P(x, y)dx - M(x, y)dy = 0$$

... now let's just make the substitution $Q(x, y) = -M(x, y)$, to obtain the form:

$$P(x, y)dx + Q(x, y)dy = 0$$

Given a differential equation in this form, we can check if it is *exact*.

Definition 2.2.1 Exact DE

A first order DE in the form:

$$P(x, y)dx + Q(x, y)dy = 0$$

Is *exact* if and only if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In the case that the DE is exact, we have a method to solve for an implicit solution.

Assuming there exists some function $u(x, y)$ which implicitly solves the DE:

$$P(x, y)dx + Q(x, y)dy = 0$$

We can use the following theorem to find it:

Theorem 2.2.1 Exactness Solutions

If $u(x, y)$ has continuous partial derivatives $u_x(x, y)$ and $u_y(x, y)$ then:

$$u(x, y) = c$$

... is an implicit solution to the DE:

$$u_x(x, y)dx + u_y(x, y)dy = 0$$

We do have to check that the function will behave properly, and actually be the solution to the DE that we want. Notably, the partial derivatives need to be continuous. Recall that $u_{xy} = u_{yx}$ for continuous partial derivatives, or more illustratively:

$$\frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial x}(u_y)$$

Which is equivalent to our condition for exactness in the definition (recall $u_x = P(x, y), u_y = Q(x, y)$):

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Now we have a new question, when solving:

$$P(x, y)dx + Q(x, y)dy = 0$$

What function $u(x, y)$ has partial derivatives of $P(x, y)$ and $Q(x, y)$... meaning we can from the following two equations to solve for the **exact** solution of the DE:

$$\frac{\partial u}{\partial x} = P(x, y)$$

$$\frac{\partial u}{\partial y} = Q(x, y)$$

We must now use the first equation to solve for $u(x, y)$:

$$\frac{\partial u}{\partial x} = P(x, y)$$

$$u(x, y) = \int P(x, y)dx$$

$$u(x,y) = M(x,y) + h(y)$$

Note that the integration constant is not just c, but $h(y)$. This is because we are integrating with respect to x and so any function of y would not change the derivative with respect to x of the anti-derivative.

Now note that we have an expression for $u(x,y)$ with an unknown integration constant and we also know that:

$$\frac{\partial u}{\partial y} = Q(x,y)$$

Compute the left hand side from what we calculated for $u(x,y)$ and set it equal to $Q(x,y)$. From this we can solve for $h(y)$

Finally, we can write the implicit function $u(x,y)$ in the form:

$$u(x,y) = c$$

... by moving all the constant to the right. If we can solve for $y(x)$ explicitly then do so.

Method to Solve a Exact DE

Given a DE in the form:

$$P(x,y)dx + Q(x,y)dy = 0$$

1. Check for exactness:
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
If true, then continue. If false, then try another method.

2. Write the equation:
$$\frac{\partial u}{\partial x} = P$$

3. Integrate both sides with respect to x. You will have some equation in the form:
$$u(x,y) = M(x,y) + h(y)$$

4. Write the equation:
$$\frac{\partial u}{\partial y} = Q$$

5. Calculate $\frac{\partial u}{\partial y}$ and substitute it into the equation above.

6. Solve for $h(y)$ by integrating $h'(y)$

Chapter 5: Linear Second Order Equations

Theory of LDEs

Definition 5.1.1 n^{th} Order LDE

A n^{th} order linear differential equation (LDE) can be written in the form:

$$a_n(x)\frac{d^{(n)}y}{dx} + a_{n-1}(x)\frac{d^{n-1}y}{dx} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Characteristics which identify an LDE:

- The dependent variable is always of first degree.
- Coefficient functions $(a_0, a_1, \ldots, a_n)$ are all functions of only the independent variable.

Definition 5.1.2 Initial Value Problems

An **initial value problem** (IVP) is a differential equation subject to:

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, ..., y^{(n-1)}(x_0) = y_{n-1}$$

Essentially this means that we limit the solutions to the differential equation to those that have the particular "initial" values. You are given all the derivative values for the "initial" time x_0.

Usually you substitute the values you know into the DE at any point and solve for constants with that assumption. From there you can just replace the constance with what you solved for and continue rearranging.

Theorem 5.1.1 Existence and Uniqueness of IVP

Given some LDE in the form:

$$a_n(x)\frac{d^{(n)}y}{dx} + a_{n-1}\frac{d^{n-1}y}{dx} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Assume $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x), g(x)$ are all continuous on some interval I, and that $a_i(x) \neq 0$ for any value of $x \in I$. Also pick some number $x_0 \in I$, then:
The LDE subject to the following initial conditions:

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, ..., y^{(n-1)}(x_0) = y_{n-1}$$

...has a **unique** solution.

This means that if you have a DE where the "coefficient functions" (the $a_n(x)$ functions) are equal to 0 at any point, then the IVP will not have a unique solution.

Definition 5.1.3 Homogeneous DE

A LDE in the form:

$$a_n(x)\frac{d^{(n)}y}{dx} + a_{n-1}(x)\frac{d^{n-1}y}{dx} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Is **homogeneous** since $g(x) = 0$

Definition 5.1.4 Non-Homogeneous DE

A LDE in the form:

$$a_n(x)\frac{d^{(n)}y}{dx} + a_{n-1}(x)\frac{d^{n-1}y}{dx} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), g(x) \neq 0$$

Is **non-homogeneous** since $g(x) \neq 0$

Theorem 5.1.2 Superposition of Solutions

Let $y_1, y_2, ..., y_k$ be solutions to a *homogeneous LDE* on I. Then any linear combination of those solutions is also a solution to that same *homogeneous LDE* on I.

Recall from linear algebra that a linear combination of things is where you scale each thing by a real number and then add them all together.

Definition 5.1.5 Wronskian

The **Wronskian** (W) of a set of functions $f_1, f_2, ...f_n$ all with $(n-1)^{th}$ order derivatives is defined as:

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Recall that the vertical bars means determinant, which works the same as the determinant of matrix with real entries, it's just now the entries are functions. Recall as well that the determinant of a 2×2 matrix is:

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Theorem 5.1.3 Non-Zero Wronskian

Let $y_1, \ldots, y_n$ be solutions to a homogeneous DE on I. Then the set of solutions is linearly independent on I if and only if:

$$W(y_1, \ldots, y_n) \neq 0$$

Definition 5.1.6 Fundamental Set

The **fundamental set** of solutions to a DE is the set of solutions which are linearly independant.

Recall that linearly independent means none are constant multiples of each other.

Theorem 5.1.4 General Solution of a HDE

If $y_1, \ldots, y_n$ is a fundamental set of a homogeneous DE, then the general solution to that DE is:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

This makes sense since we know that a linear combination of solutions to a homogeneous is itself a solution, and since the fundamental set is linearly independent, combining all the of the functions in a linear combination gives you the most general form of the solution.

Theorem 5.1.5 Solutions to a Non-HDE

Given some non-homogenous DE in the form:

$$a_n(x)\frac{d^{(n)}y}{dx} + a_{n-1}(x)\frac{d^{n-1}y}{dx} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), g(x \neq 0)$$

We can find the general solution of it by finding the general solution to the corresponding homogenous DE:

$$a_n(x)\frac{d^{(n)}y}{dx} + a_{n-1}(x)\frac{d^{n-1}y}{dx} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Say $y_1, \ldots, y_n$ is a fundamental set of the homogeneous DE on I and y_p is any particular solution to the non-homogenous DE on I, then:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$$

Is the general solution to the non-homogeneous DE on I.

What this means is that if we have a non-homogeneous LDE, then we can find the general solution to it by finding the general solution to the homogeneous LDE (called the *complementary solution* y_c) and add on any particular solution to the non-homogeneous LDE. Every second order LDE has two families of solutions, and so the general form will have two terms at least (if it is homogeneous).

Theorem 5.1.6 Abel's Theorem

A second order LDE in the form:

$$y'' + P(x)y' + Q(x)y = 0$$

...has two solutions families y_1, y_2. Say you found one solution family y_1 through some method, then the other solution family is:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

The previous theorem is also called *reduction of order*.

Solving Second Order LDEs

If you want to solve a homogeneous LDE, then we need to begin with a simple case, constant coefficients:

$$ay'' + by' + cy = 0$$

Let's take a guess and say $y = e^{mx}$ is a solution to this DE. If we can solve for m in terms of a, b, c then this really is a solution.

$$y = e^{mx}$$
$$y' = me^{mx}$$
$$y'' = m^2 e^{mx}$$

Substituting that into our LDE:

$$ay'' + by' + cy = 0$$
$$a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$$
$$e^{mx}(am^2 + bm + c) = 0$$

This tells us that:

$$e^{mx} = 0$$
$$am^2 + bm + c = 0$$

The second to last equation has no solutions, the last equation has two solutions of:

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

And so not only was our guess correct, but it gave us ideally BOTH of our solutions:

$$ay'' + by' + cy = 0 \implies y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$$

There are three cases for how this can be written:

1. $m_1 \neq m_2,$ and $m_1, m_2 \in \mathbb{R}$

   In this case our general solution is just a linear combination of the two solutions since they are linearly independent:

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

2. $m_1 = m_2 = m = \frac{-b}{2a}$

   In this case, the two solutions are not linearly independent and so you have only found one of the two solution families, namely: $y_1 = e^{mx}$. You can find the other through *reduction of order*: Applied to the equivalent DE:

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int b\frac{a}{d}x}}{y_1^2} dx$$

   Using the fact that:

$$m = \frac{-b}{2a} \implies 2m = \frac{-b}{a}$$

$$y_2 = e^{mx} \int \frac{e^{2mx}}{e^{2mx}} dx$$

$$y_2 = e^{mx} \int dx$$

$$y_2 = xe^{mx}$$

   Now we have two linearly independent families of solutions and so the general solution is:

$$\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$$

3. $m_1, m_2 \in \mathbb{C}$

   In this case:

$$m_{1,2} = \alpha \pm \beta i$$

   These are distinct and so you could write the general solution as:

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$

   However this can be written in a simpler way without complex numbers as:

$$\boxed{y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))}$$

This method extends as you'd expect to higher order LDEs.

Solving Non-Homogeneous LDEs with Constant Coefficients

Now that we know how to find the solution to the homogeneous DE, we know that we can find the general solution to the non-homogeneous DE by:

$$y = y_c + y_p$$

As in the general solution to the non-homogeneous DE is the general solution tot he homogeneous complementary DE, plus some particular solution to the non-homogeneous DE. We can find a particular solution based on $g(x)$ (the function after the equal sign). In the following table all capital letters are unknowns to be solved for, lower case letters are to identify form.

$g(x)$	**Guess for** $y_p(x)$
a	A
$ax + b$	$Ax + B$
$ax^2 + bx + c$	$Ax^2 + Bx + C$
$\vdots$	$\vdots$
$a_n x^n + \cdots + a_1 x_1 + a_0$	$A_n x^n + \cdots + A_1 x_1 + A_0$
$\sin(\alpha x)$	$A\cos(\alpha x) + B\sin(\alpha x)$
$\cos(\alpha x)$	$A\cos(\alpha x) + B\sin(\alpha x)$
$e^{\alpha x}$	$Ae^{\alpha x}$
$(ax + b)e^{\alpha x}$	$(Ax + B)e^{\alpha x}$
$(ax^+bx + c)e^{\alpha x}$	$(Ax^2 + Bx + C)e^{\alpha x}$

If $g(x)$ can be factored into some of the forms above, then your guess would be the product of each corresponding guess for each factor.

Make the substitution $y =$ "your guess" and then solve for the constants.

The only complication comes if the general solution to the homogeneous DE has some multiple of the particular solution in it. In this case you need to multiply the particular solution by x as many times as needed to make it linearly independent from y_c.

Additionally, if your function $g(x)$ cannot be written in the forms mentioned above, then you must use the following method,

Variation of Parameters

This section defines a new method to solve higher order non-homogenous LDE. This method does not require constant coefficients, however since it does require us to find the homogenous solution for our examples at the time we will still have constant coefficients.

This method is particularly convenient when To find the general solution to:

$$y'' + Ay' + By = f(x)$$

Write it in the form:

$$y = y_c + y_p$$

Where y_c is the general solution to the related homogenous DE written as:

$$y_c = c_1 y_1 + c_2 y_2$$

... then by the method of **variation of parameters** the particular solution would be:

$$y_p = u_1 y_1 + u_2 y_2$$

... where:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W(y_1, y_2)}$$

This is Cramer's rule applied to solve the following system:

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = 0$$

Chapter 8: Laplace Transforms

8.1 The Laplace Transform

This section begins with a definition of a transform:

Definition 8.1.1 Transform

A transform is a function which takes in functions as inputs, and returns functions as output.

The derivative (also called the derivative operator) is a transform because you can apply it to a function and you get a function back out as a result:

$$\frac{d}{dx} : x^2 \mapsto 2x$$

The *Laplace Transform* is another example of a transform, which is defined by the following:

Definition 8.1.1 The Laplace Transform

The **Laplace Transform** of some function $f(t)$ is:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

As you can see this fits our definition of a transform as you can put $f(t)$ into the transform and you get another function out of it. The Laplace Transform is useful in solving IVPs, simplifying the process to solving an algebraic equation.

Often when you compute the Laplace Transform of a function, you need to put a restriction on s to make the limit converge, since this is an improper integral. Also note that the output function is a function of s since the function gets evaluated at specific values of t.

One of the most important properties of the Laplace transform is that it is linear, meaning:

Theorem 8.1.2 Linearity of Laplace

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\}$$

... for $\alpha, \beta \in \mathbb{R}$

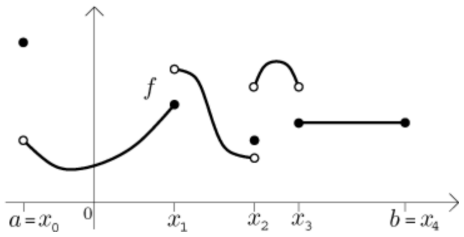
The following is a table of a few common Laplace Transforms which can be derived using the definition:

- $$\mathscr{L}\{1\} = \frac{1}{s}$$
- $$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ for } n \in \mathbb{N}$$
- $$\mathscr{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$$
- $$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$
- $$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

Next we want to understand when the Laplace transform of a function exists or doesn't. For that we need to the following two definitions:

Definition 8.1.2 Piecewise Continuous

A function is **piecewise continuous** on $[0, \infty)$ if for any interval you choose in $[0, \infty)$ contains a finite amount of finite discontinuities, and the function is continuous otherwise.



Definition 8.1.3 Exponential Order

A function f is said to be of **exponential order** if there exists constants c, $M > 0$, $T > 0$ such that:

$$|f(t)| \leq Me^{ct}$$

... for all $t > T$.

Essentially if your function is increasing but grows slower than an exponential function, it is of exponential order.

Theorem 8.1.2 Existance of $\mathscr{L}\{f(t)\}$

If $f(t)$ is peicewise continuous on the interval $[0, \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

Notice that the previous theorem does not state that these are the *only* conditions for the existence of the transform, only that they are sufficient to guarantee it. If a function does not contain these properties it's transform may or may not exist, but this theorem tells us that if a function *does* have these properties, then its transform *definitely* exists.

8.2 The Inverse Laplace Transform

Definition 8.2.1 The Inverse Laplace Transform

If:

$$\mathscr{L}\{f(t)\} = F(s)$$

... then:

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

... where $\mathscr{L}^{-1}\{F(s)\}$ is the inverse Laplace transform of $F(s)$.

To find the inverse Laplace of a function you want to match it to the forms of the regular Laplace transform on page 17. This can be done by algebraic techniques like partial fraction decomposition. It is also useful to know:

Theorem 8.2.1 Linearity of the Inverse Laplace Transform

The Inverse Laplace Transform is a linear transform, meaning:

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\}$$

... for $\alpha, \beta \in \mathbb{R}$

Laplace of Derivatives

The Laplace transform is useful in solving initial value problems, in this section we will see exactly how and why.

Let's say we want to evaluate the Laplace of the first derivative of f instead of the function itself:

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

... by integration by parts:

$$= [e^{-st} f(t)]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\mathscr{L}\{f(t)\}$$

Meaning:

$$\boxed{\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)}$$

As you can see, we have related the Laplace of the derivative of a function to the function itself, and its Laplace. Similarly we can see:

$$\boxed{\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)}$$

$$\boxed{\mathscr{L}\{f'''(t)\} = s^3\mathscr{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)}$$

Or generally:

Theorem 8.2.2 Laplace of the Derivative

If $f, f', \ldots, f^{(n-1)}$ are all continuous on $[0, \infty)$ and are of exponential order, and if $f^{(n)}$ is piecewise continuous on the same domain, then:

$$\mathscr{L}\{f^{(n)}(t)\}s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

This is useful because we can turn all the derivatives of a function into the Laplace transform minus some initial values. This, along with the linearity of the Laplace transform make it a very effective technique to solve IVPs. The following is a fully worked example demonstrating this method:

Example 8.2.1 Laplace to Solve an IVP

Solve the following IVP using the Laplace Transform:

$$y'' - 3y' + 2y = e^{-4t}, y(0) = 1, y'(0) = 5$$

Begin with your equation:

$$y'' - 3y' + 2y = e^{-4t}$$

Take the Laplace of both sides:

$$\mathscr{L}\{y'' - 3y' + 2y\} = \mathscr{L}\{e^{-4t}\}$$

Use the linearity of the Laplace Transform:

$$\mathscr{L}\{y''\} - 3\mathscr{L}\{y'\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

Apply theorem 8.2.2 which is how the Laplace transform relates to derivatives:

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \mathscr{L}\{e^{-4t}\}$$

Apply the Laplace transform to the exponential:

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s+4}$$

Substitute the values we know:

$$s^2Y(s) - s - 5 - 3(sY(s) - 1) + 2Y(s) = \frac{1}{s+4}$$

Solve for $Y(s)$ and apply PFD:

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} = \frac{-16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

Apply the inverse Laplace transform to both sides:

$$\mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\{\frac{-16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}\}$$

Left side becomes the solution to the IVP:

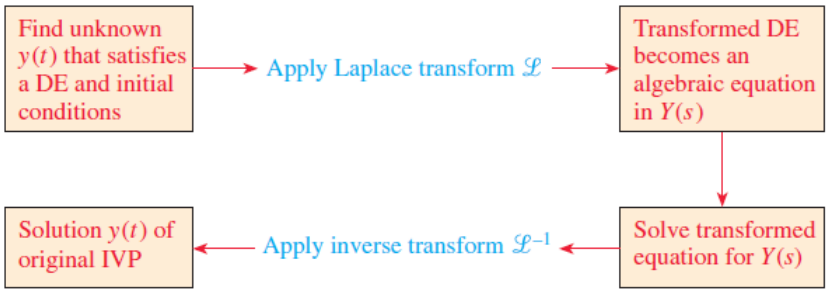
$$y = -\frac{16}{15}\mathscr{L}^{-1}\{\frac{1}{s-1}\} + \frac{25}{6}\mathscr{L}^{-1}\{\frac{1}{s-2}\} + \frac{1}{30}\mathscr{L}^{-1}\{\frac{1}{s+4}\}$$

Match the forms to find the inverse:

$$y = -\frac{16}{15}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Which is our solution to the IVP.

The general method can be described by this graphic:



8.4 Translation Theorems

In this section we investigate the effect of multiplying the input to the transform by two specific functions. Another way to think about this is we are investigating:

$$\mathscr{L}^{-1}\{F(s-a)\}$$
$$\mathscr{L}^{-1}\{e^{-as}F(s)\}$$

These theorems can help save time instead of having to apply things like IBP multiple times.

Theorem 8.4.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}$, then:

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

In a sense, multiplying the input of the transform by e^{at} horizontally shifts the Laplace transform right by a units. The proof of this is by definition:

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

Example 8.4.1 First Translation Theorem

Compute:

$$\mathscr{L}\{e^{5t}t^3\}$$

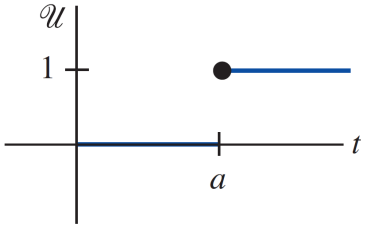
$$\mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}\ |_{s\to s-5} = \frac{3!}{s^4}\ |_{s\to s-5} = \frac{6}{(s-5)^4}$$

Often functions are either *on* or *off*. To turn a function *on* you can multiply it by 1, and to turn it *off* you can multiply it by 0. If we want to turn the function $f(t)$ on at time $t = 4$, then we want a function $\mathscr{U}(t)$ that implements the concept: *"Output 0 until $t = 4$, and then output 1"*... which we could multiply by $f(t)$, as in $f(t)\mathscr{U}(t)$. The function which implements this is called the *unit step function* or the *heaviside function*, and would be written in this example as $\mathscr{U}(t-4)$.

Definition 8.4.1 The Unit Step Function

The unit step function $\mathscr{U}(t-a)$ is defined as:

$$\mathscr{U}(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$



Note that at $t = a, \mathscr{U} = 1$.

The unit step function is also useful in turning piecewise functions into direct equation form. Using the following theorems relations:

1. Two case piecewise:

$$f_2(t) = \begin{cases} g(t) & 0 \leq t < a \\ h(t) & t \geq a \end{cases}$$

$$f_2(t) = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

2. Three case piecewise, first and last 0:

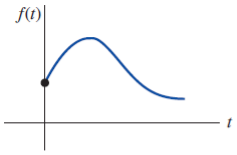
$$f_3(t) = \begin{cases} 0 & 0 \leq t < a \\ g(t) & a \leq t < b \\ 0 & t \geq a \end{cases}$$

$$f_3(t) = g(t)[\mathscr{U}(t-a) - \mathscr{U}(t-b)]$$

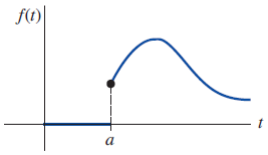
This brings us to our second transition theorem. While the first one was concerned with horizontally shifting the output of the transform a units to the right, this theorem is concerned with moving the *input* function to the right by a units with the additional condition that the function should be turned off until $t = a$, meaning the input would be transformed to:

$$f(t-a)\mathscr{U}(t-a)$$

Visually that is:



(a) $f(t), t \geq 0$



(b) $f(t-a)\mathcal{U}(t-a)$

Theorem 8.4.2 Second Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}^+$, then:

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Additionally these is a way to write this theorem in the a more useful form for when f is not in the correct shifted form:

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}}$$

The Derivative of the Laplace Transform

In this section we'd like to evaluate the derivative of the Laplace transform as in:

$$\frac{d}{ds}\mathscr{L}\{f(t)\}$$

... we begin by definition:

$$= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)dt$$

$$= -\int_0^\infty e^{-st}f(t)dt$$

$$= -\mathscr{L}\{tf(t)\}$$

This can be generalized to:

Theorem 8.4.3 Derivative of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

This is useful in examples like the following:

Example 8.4.2 Derivative of the Laplace Transform

Compute:

$$\mathscr{L}\{t\sin(kt)\}$$

$$\mathscr{L}\{t\sin(kt)\} = -\frac{d}{ds}\mathscr{L}\{\sin(kt)\} = -\frac{d}{ds}\left(\frac{k}{s^2+k^2}\right) = \frac{2ks}{(s^2+k^2)^2}$$

8.6 Convolution

This section is motivated by a question like the following:

$$\text{Evaluate: } \mathscr{L}^{-1}\{\frac{1}{s-1}\cdot\frac{1}{s^2+1}\} = \mathscr{L}^{-1}\{\mathscr{L}\{e^t\}\mathscr{L}\{\sin(t)\}\}$$

This is not achieved by taking the product of the arguments, but rather we need a new operation.

In this section we define a new operation called *convolution* which can be applied to two functions. Convolution is *not* multiplication however we do use a $*$ to denote it.

Definition 8.5.1 Convolution

The convolution $(*)$ of two functions f and g is defined as:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

The reason this is of any use to us is because of the following theorem:

Theorem 8.5.1 Convolution Theorem

If $f(t)$ and $g(t)$ are piecwise continuous on $[0, \infty)$ and of exponential order, then:

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$$

This can be used in reverse:

$$f * g = \mathscr{L}^{-1}\{F(s)G(s)\}$$

... to answer a question like the one stated at the beginning of this section:

$$\mathscr{L}^{-1}\{\mathscr{L}\{e^t\}\mathscr{L}\{\sin(t)\}\} = e^t * \sin(t)$$

Chapter 11: Fourier Series

This chapter is all about the Fourier Series which is an analytical tool used to help solve partial differential equations. This chapter is just the theory of the Fourier Series and how to find the Fourier Series of a function, we will not go into it's applications.

The Fourier Series (FS) is a series just like the Taylor Series. It is a way to write the function as an infinite sum. While the Taylor Series came from our study of Power Series, the FS comes from Linear Algebra. First we recall some of those topics.

Recall that a *vector* is an element of a special kind of set called a Vector Space. One such vector spaces is $\mathbb{R}^n$ which is where we get our visual understanding of vectors in space.

Vector spaces can also have something called an *inner product* which is a way to combine vectors with a certain list of special properties. The exact list is not important at this moment, but all inner product have the same list. In $\mathbb{R}^n$ the inner product is the dot product, but in other vector spaces the inner product is different. The inner product of two vectors is denoted:

$$< \vec{v_1}, \vec{v_2} >$$

Recall that in $\mathbb{R}^3$ we define two vectors to be **orthogonal** is they have a dot product of *zero*.

Now let us consider a more abstract vector space, a vector space of functions. This vector space passes all the tests of a vector space like we learned at the end of MTH 141. This makes functions vectors just like arrows in space. We would like to also have an inner product (like a dot product) for this vector space:

Definition 11.1.1 Inner Product of Functions

The inner product of two function f_1 and f_2 on the interval $[a, b]$ is:

$$< f_1, f_2 >= \int_a^b f_1(x) f_2(x) dx$$

The following list of properties is true regarding the inner product of functions f, g, h and a constant k:

1. $$< f, g >=< g, f >$$

2. $$< kf, g >= k < f, g >$$

3. $$< f, f >= 0$$

   ... only if $f = 0$, otherwise the result is positive.

4. $$< f + g, w >=< f, w > + < g, w >$$

Definition 11.1.2 Orthogonal Functions

Two function are **orthogonal** on $[a, b]$ if their inner product is equal to 0 on $[a, b]$, as in (for functions f, and g):

$$< f, g >= \int_a^b f_1(x) f_2(x) dx = 0$$

Definition 11.1.3 Orthogonal Sets

A set of function is said to be orthogonal on $[a, b]$ if the inner product between any two elements is 0 on $[a, b]$.

A set is **orthonormal** if all the elements of the set also have a length of 1, length (also called **norm**) is defined next:

Definition 11.1.4 Norm of a Vector

The norm of a vector $\vec{v}$ is:

$$\left\|\vec{v}\right| = \sqrt{<\vec{v}, \vec{v}>}$$

This above definition also applies to functions. A norm is like the length of a vector, but we want to generalize this concept to things that done have a length like functions.

More on Vectors

Remember given some vector $\vec{v}$ in $\mathbb{R}^n$ we can write $\vec{v}$ as a linear combination of orthonormal basis vectors. Given the basis:

$$\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$$

We can write any vector v as:

$$\vec{v} = a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n}$$

... all you have to do is figure out the sequence a_n.

Our goal in a FS is to do the exact same thing with functions. Given the orthonormal set:

$$\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{n\pi x}{p}\right)\}$$

... we can show that it is both *orthonormal* and *spans* all periodic functions with a period of p. Notice also it is an infinite set since n is some integer. Given that this is true, this means that any such periodic function can be written as a linear combination of the functions in the set, as in:

$$f(x) = k(1) + a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)$$

... all we have to do is figure out k, a_n, and b_n. The equation above can be rewritten as:

$$f(x) = k(1) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

The coefficients k, a_n, and b_n turn out to be:

Definition 11.1.5 Fourier Series

The **Fourier Series** of a function f defined on the symmetrical interval $(-p, p)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi}{p}) + b_n \sin(\frac{n\pi}{p}) \right)$$

... where:

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos(\frac{n\pi}{p}x)dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin(\frac{n\pi}{p}x)dx$$

Now the most important thing is that this series actually converges to the function wherever possible. Our function f doesn't have to be fully continuous, just piecewise continuous. At the discontinuities the FS converges to the midpoint.

Theorem 11.1.1 Convergence of the Fourier Series

If f is piecewise smooth on $[-L, L]$ then the Fourier Series of f:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi}{p}) + b_n \sin(\frac{n\pi}{p}) \right)$$

converges for all x on $[-L, L]$, in fact it converges to exactly:

$$F(x) = \begin{cases} f(x) & -L < x < L \text{ and } f \text{ is continuous at } x \\ \frac{f(x^+)+f(x^-)}{2} & -L < x < L \text{ and } f \text{ is discontinuous at } x \\ \frac{f(-L^+)+f(L^-)}{2} & x = L \text{ or } x = -L \end{cases}$$

If the function happens to be even or odd, then the formulas simplify. Recall that a function can be:

$$\text{Even: } f(x) = f(-x)$$

$$\text{Odd: } f(-x) = -f(x)$$

Additionally there are some useful properties of even and odd functions:

1. $$even \cdot even = even$$

2. $$odd \cdot odd = even$$

3. $$even \cdot odd = odd$$

4. $$even - even = even$$

5. $$odd - odd = odd$$

6. If f is *even*:
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

7. If f is *odd*:
$$\int_{-a}^{a} f(x)dx = 0$$

Definition 11.1.6 Cosine or Sine Fourier Series

The FS of an **even** function (f) on $(-p, p)$ is called the **cosine series**:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right)$$

... where:

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)dx$$

The FS of an **odd** function (f) on $(-p, p)$ is called the **sine series**:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right)$$

... where:

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)dx$$

Half-Range Extensions

The FS is defined on $(-p, p)$ but in practice a lot of the time you only want the FS of the function on $(0, L)$. All you have to do in this case is adjust your choice of p to use the regular FS:

Function	**Choice of p**
Cosine Series on $(0, L)$	$p = L$
Sine Series on $(0, L)$	$p = L$
Fourier Series on $(0, L)$	$p = \frac{L}{2}$

There is a lot more to FS than what was discussed here, I recommend reading $12.1 - 12.3$ from the textbook *Advanced Engineering Mathematics* and watching *Dr. Trefor Bazett* on youtube to get more depth to this topic.

Chapter 1: Parametric Equations and Polar Coordinates

1.1: Parametric Equations

A parametric equation is a way to write a relation between x and y as a function of some other variable t (usually thought of as time), even if the graph of this relation is not a function.

The idea is that every point on the graph (x, y) is assigned a particular t value, and so we think of both coordinated as functions of t as in $(x(t), y(t))$.

Definition 1.1.1 Parametric Equations

If x and y are continuous functions of t on an interval I, then the equations:

$$x = x(t)$$

$$y = y(t)$$

... are called **parametric equations** with t as a parameter. The graph of parametric equations is called a parametric curve, or plane curve C.

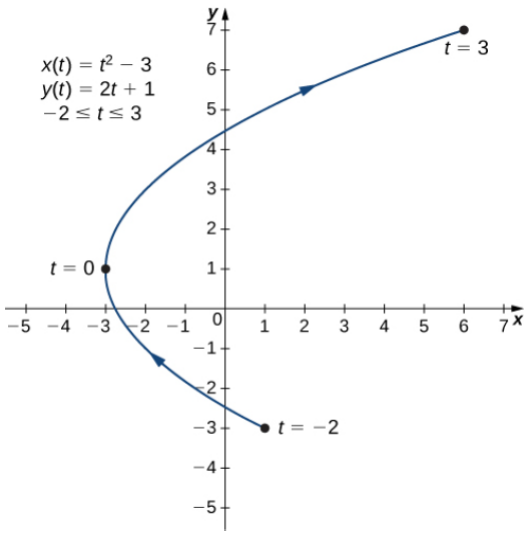
For example if you graph the following parametric equations:

$$x(t) = t^2 - 3$$

$$y(t) = 2t + 1$$

$$-2 \leq t \leq 3$$

... you get the following graph:



Notice that we had to limit the range of t, this made it so that the graph has a start and end point. Also notice that the graph is not a function even through x and y are both functions of t. Finally also notice that the graph has an orientation, as t increases the curve is drawn bottom to top.

Now we discuss converting between parametric and rectangular form.

Parametric to Rectangular

This process is called *eliminating the parameter*.

1. Isolate t in one or both equations.
2. Substitute one into the other.
3. Solve for y if possible.

Rectangular to Parametric

This process is called *parametrization of the curve*. There is an infinite number of ways to parametrize a rectangular curve. In general the process is:

1. Choose some function $f(t)$ and set $x(t) = f(t)$.
   (a) $f(t)$ must have a range of $f(t) \in \mathbb{R}$.
2. Substitute $f(t)$ into the equation for x.
3. Solve for $y(t)$.

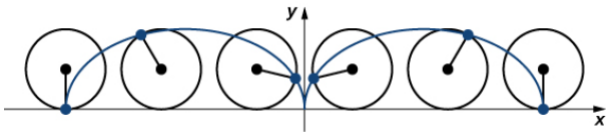
Definition 1.1.2 Cycloids

A cycloid is a type of parametric equation in the form:

$$x(t) = a(t - sin(t))$$

$$y(t) = a(1 - cos(t))$$

Cycloids can be thought of as the curve traced out by a point on a wheel:



1.2 Calculus of Parametric Curves

We begin this section with a definition of the derivative in parametric form:

Definition 1.2.1 Parametric Derivative

Given a plane curve C defined by $x = x(t)$ and $y = y(t)$, where $x'(t)$ and $y'(t)$ both exist, the derivative is then given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

This will return a function of t, which will output the slope of the tangent at any point $(x(t), y(t))$.

Higher order derivatives are a little harder to define, so we just need to additionally know that the section derivative is defined as:

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt}$$

Next we deal with integration:

Definition 1.2.2 Parametric Integration

Given some non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), y = y(t), a \leq t \leq b$$

... and assuming $x(t)$ is differentiable, the area under the curve is given by:

$$A = \int_a^b y(t)x'(t)dt$$

We can also apply integration to find the arc length of a curve over some range of t:

Definition 1.2.3 Parametric Arc Length

Given some plane cirve defined by parametric equations:

$$x = x(t), y = y(t), t_1 \leq t \leq t_2$$

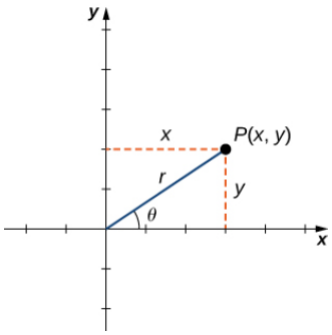
... and assuming that $x(t)$ and $y(t)$ are both differentiable, the arc length is given by:

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

1.3 Polar Coordinates

In rectangular coordinates, we can assign an ordered pair (x, y) to every point in the grid based on its distance along the $x-axis$ and distance along the $y-axis$ respectively. Polar coordinates as an alternative way of assigning an ordered pair to every point in the grid.

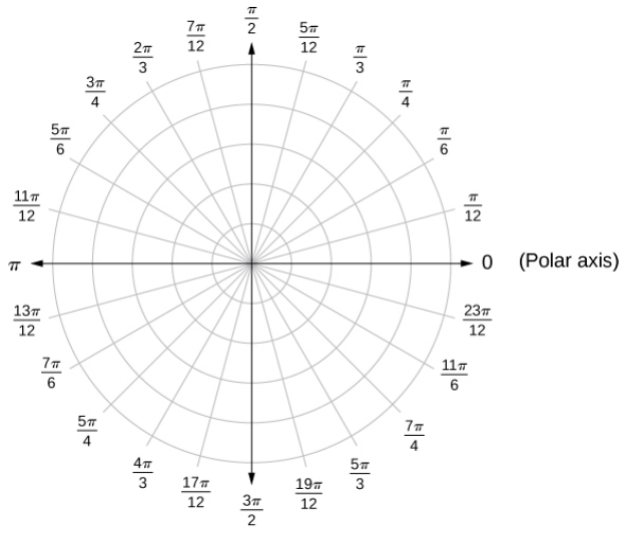
Using polar coordinates, we can convert a point (x, y) into a new ordered pair (r, θ) using the following conversion:



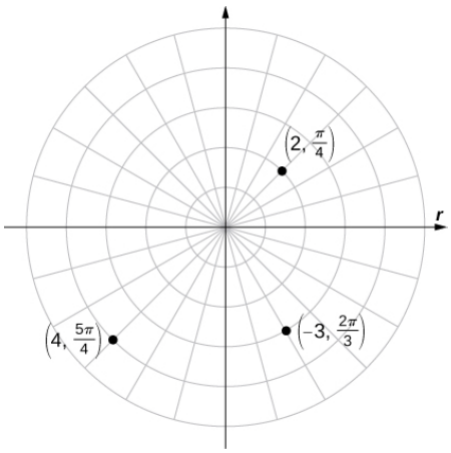
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$

It is good to imagine a ray coming from the origin of length r and making an angle θ with the positive $x - axis$ and drawing the point at it's tip. This should remind you of a vector, and for good reason.

To make our lives easier, we sometimes choose to label the grid as:



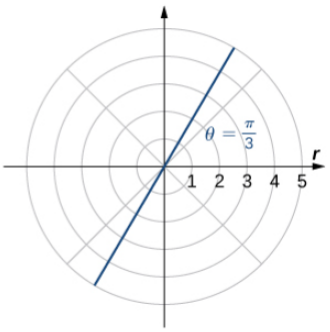
The origin is called the **pole** in this context, and the horizontal axis is the **polar axis**. The innermost circle denotes a radial distance of 1. We can see a few points plotted on the following graph:

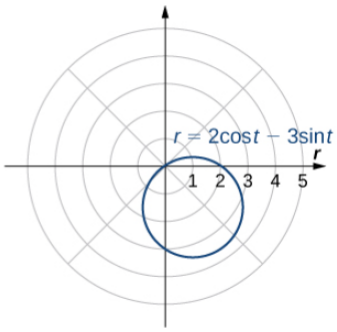
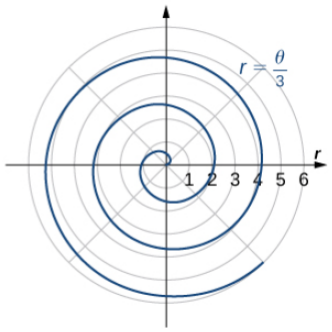
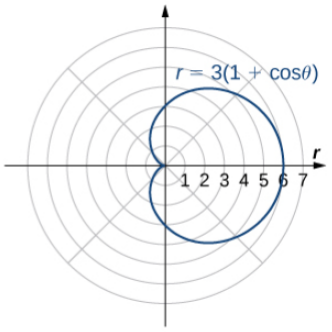


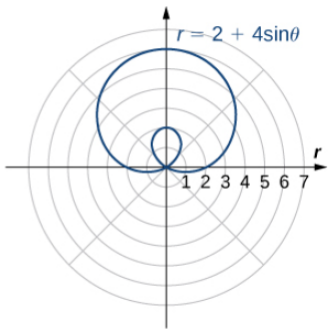
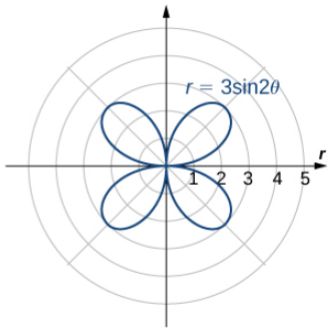
We should note that the polar plane is exactly the same geometrically as the Cartesian plane, but we are labelling things in the polar context.

Polar curves are created when we define a function in the form $r = f(\theta)$. Meaning the **length** of a ray is a function of the **angle** it makes with the positive $x-axis$. Graphing polar functions may seem confusing at first, because we have little experience with these types of functions. As you practice you will begin to see patterns. Imagine being asked to sketch $f(x) = e^x x^2$ without learning what the graphs of e^x or x^2 are.

The following is a list of graphs and their plots.

Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	

Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	

Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

In rectangular coordinates we considered the symmetry of a function by calling it even and odd. We have similar definitions in polar coordinates:

Definition 1.3.1 Symmetry in Polar Curves

There are three types of symmetry in polar curves:

1. **Symetry about the Polar axis**: Either:

$$f(\theta) = f(-\theta)$$

Or:

$$f(\theta) = -f(\pi - \theta)$$

2. **Symmetry about the Pole**: Either:

$$f(\theta) = f(\theta + \pi)$$

Or:

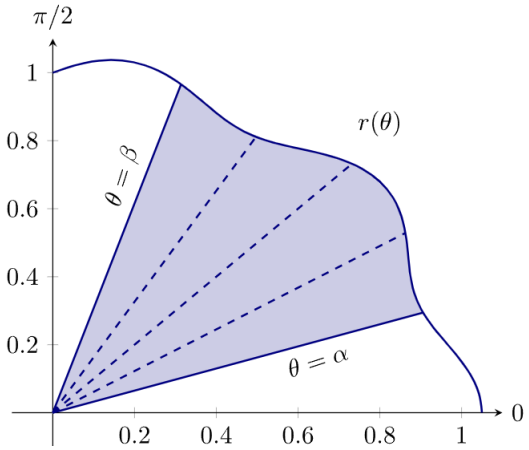
$$f(\theta) = -f(\theta)$$

3. **Symmetry Across the line** $\theta = \frac{\pi}{2}$:

$$f(\theta) = f(\pi - \theta)$$

1.4 Area and Arc Length in Polar Coordinates

Given some polar plot, we may be interested in measuring the area enclosed by some part of it. Just like in rectangular coordinates where we used an integral ($\int dx$) to solve for this area, in polar coordinates we use an integral ($\int d\theta$). The bounds of the integral are now angles, and the represents what rays bound the integral, as in the following diagram:



We integrate all the lengths of the rays which produce the polar plot into an area using the following definition:

Definition 1.4.1 Polar Integration

Suppose f is continuous and non-negative on the interval $\alpha \leq \theta \leq \beta$ and that interval has a length between 0 and 2π (inclusive of 2π). Then the area of the region bounded by $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

For the area between two curves $r_1 = f(\theta)$ and $r_2 = g(\theta)$ (where informally $r_1 > r_2$) we use:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta$$

... and α, β represent the angles of their points of intersection.

Finally we define a formula for the arc length of a polar function:

Definition 1.4.2 Polar Arc Length

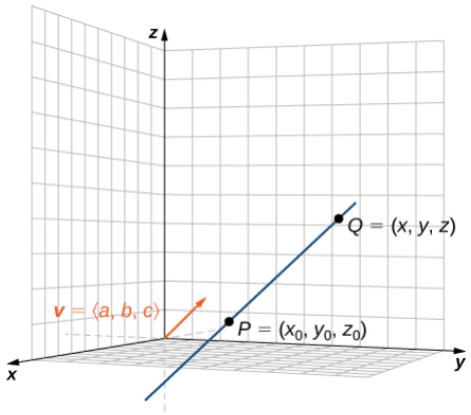
Let f be a function with a continuous derivative on an interval $\alpha \leq \theta \leq \beta$. The arc length of the graph from $\theta = \alpha$ to $\theta = \beta$ is:

$$L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Chapter 2: Vectors in Space

2.5: Equations of Lines & Planes in Space

The vector equation of a line is one way to describe a line in 3D space. For it you need a point on the line, and a direction vector from that point, and all other points on the line will be in the direction (or opposite) of that vector in space.



In general this can be written as:

$$\vec{x} = \vec{P_0} + t\vec{d}$$

... where:

1. $\vec{x}$ are the coordinates to any point on the line.
2. $\vec{P_0}$ is your initial point.
3. $\vec{d}$ is your direction vector.
4. t is an arbitrary scaling factor.

You can then then separate the components of the vector to get the parametric form of a line:

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = y_0 + tc$$

... assuming $\vec{d} = [a, b, c]^T$. You can then eliminate the parameter to get the **symmetric form** of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

A plane can be written by using the general idea that all vectors in the plane are parallel to the normal. Say you have some starting point $\vec{P_0}$, and a normal $\vec{n}$, then the vector $(\vec{PQ})$ which connects the starting point to any point on the plane Q would be orthogonal to the normal, as in:

$$\vec{n} \cdot \vec{PQ} = 0$$

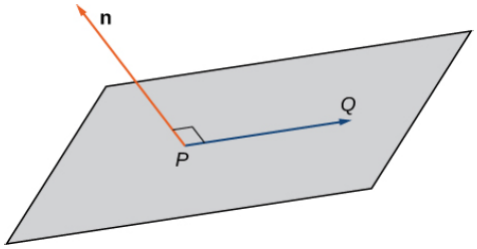
$$[a, b, c]^T \cdot [x - x_0, y - y_0, z - z_0]^T = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the scalar equation of the plane. Sometimes we collect constants are rewrite it as:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

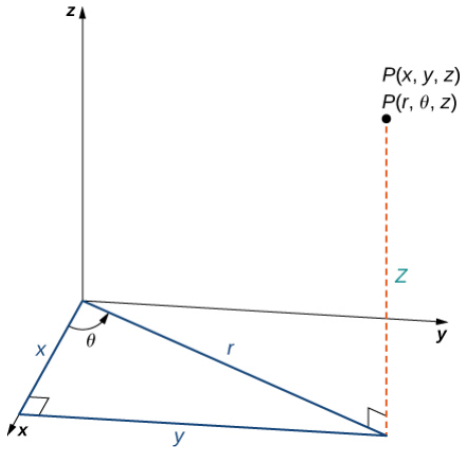
This just makes the equation a little easier to use since the coefficients on the left are just the normal vector, and the values on the right can just be calculates by being given a point on the plane.



If you are given three points on the plane and no normal, you can find the normal by taking the cross product of any two of the vectors between two points, and then using that as the normal for the above process (picking a random point that was given to be P_0).

2.7: Cylindrical & Spherical Coordinates

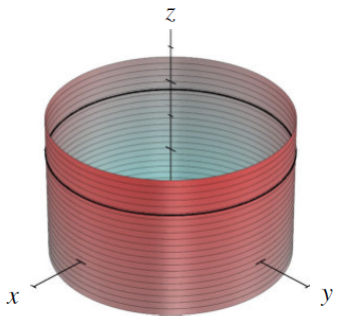
In cylindrical coordinates, a point in $3D$ space is given by an ordered triple (r, θ, z) where:



1. r is the radial distance on the $xy-$plane.
2. θ is the angle that radial line makes with the positive $x-$axis.

3. z is the standard coordinate off the $xy-$plane.

It is called cylindrical coordinates because given some radial distance, you construct a cylinder until you know what θ and z are:



Given some rectangular coordinate (x, y, z), and the cylindrical coordinates of the same point (r, θ, z), then they are related in the following equations:

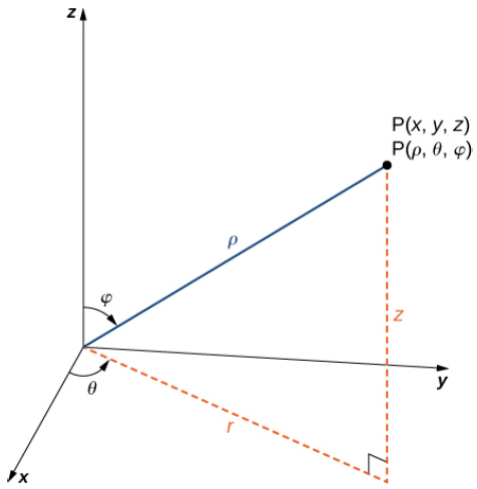
Cylindrical to Rectangular

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

Rectangular to Cylindrical

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$z = z$$

In **spherical coordinates**, a point in 3D space is given by an ordered triple (ρ, θ, ϕ).



Where:

- ρ is the distance between the point and the origin.
- θ is the angle the radial line makes with the positive $x-$axis.
- ϕ is the angle formed by the positive $z-$axis and the line segment $\bar{OP}$.

Given some point (ρ, θ, ϕ) it can be related to Rectangular and Cylindrical coordinates using the following:

Spherical to Rectangular

$$x = \rho \sin(\phi) \cos(\theta)$$
$$y = \rho \sin(\phi) \sin(\theta)$$
$$z = \rho \cos(\phi)$$

Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Spherical to Cylindrical

$$r = \rho \sin(\phi)$$
$$\theta = \theta$$
$$z = \rho \cos(\phi)$$

Cylindrical to Spherical

$$\rho = \sqrt{r^2 + z^2}$$
$$\theta = \theta$$
$$\phi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$

Chapter 3: Vector-Valued Functions

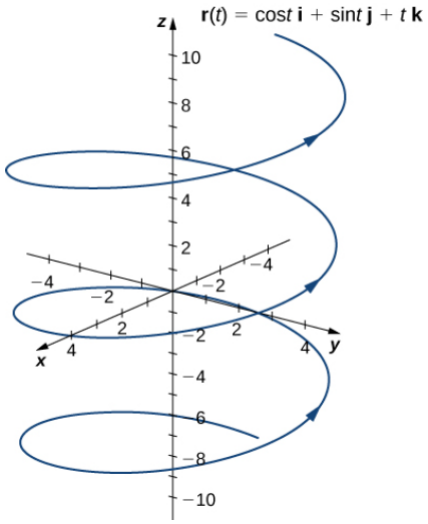
3.1 Vector-Valued Functions (VVFs)

Definition 3.1.1 Vector-Valued Functions

A **vector-valued function** is a function in the form:

$$\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

Where each f, g, h are parametric functions of t. Plotting VVFs is very similar to plotting polar functions.



Definition 3.1.2 Vector-Valued Limits

A vector valued function $\vec{r}(t)$ approaches the limit $\vec{L}$ as t approaches a,

$$\lim_{t\to a} \vec{r}(t) = L$$

... given that:

$$\lim_{t\to a} \left|\left| \vec{r}(t) - \vec{L} \right|\right| = 0$$

Theorem 3.1.1 Calculating the Limit of a VVF

Let f, g, and h be functions of t. Then:

$$\lim_{t\to a} \vec{r}(t) = \begin{bmatrix} \lim_{t\to a} f(t) \\ \lim_{t\to a} g(t) \\ \lim_{t\to a} h(t) \end{bmatrix}$$

Provided all the limits exist.

We can then define the continuity of a vector valued function:

Definition 3.1.3 Vector-Valued Continuity

A vector valued function $\vec{r}(t)$ is continuous at $t = a$ if the following three conditions hold:

1. $$\vec{r}(a) \text{ exists.}$$
2. $$\lim_{t \to a} \vec{r}(t) \text{ exists.}$$
3. $$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

3.2 Calculus of VVFs

We begin with the formal definition of the derivative of a VVF, and then the method we actually use to compute the derivatives:

Definition 3.2.1 VVF Derivative

The derivative of $\vec{r}(t)$ is:

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

We however never actually compute the derivative using this method, instead we use the following theorem:

Theorem 3.2.1 Calculating the Derivative of a VVF

Let f, g, and h be differentiable functions of t. Then:

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$$

In general, the derivative operator works the same as with regular non-vector valued functions. The following is an explicit list of rules:

1. $$\frac{d}{dt}[c\vec{r}(t)] = c\frac{d}{dt}[\vec{r}(t)]$$

2. $$\frac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] = \frac{d\vec{r}(t)}{dt} \pm \frac{d\vec{u}(t)}{dt}$$

3. $$\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\frac{d\vec{u}(t)}{dt}$$

4. $$\frac{d}{dt}[\vec{r}(t) \cdot \vec{u}(t)] = \frac{d\vec{r}(t)}{dt} \cdot \vec{u}(t) + \vec{r}(t) \cdot \frac{d\vec{u}(t)}{dt}$$

5. $$\frac{d}{dt}[\vec{r}(t) \times \vec{u}(t)] = \frac{d\vec{r}(t)}{dt} \times \vec{u}(t) + \vec{r}(t) \times \frac{d\vec{u}(t)}{dt}$$

6. $$\frac{d}{dt}[\vec{r}(f(t))] = \frac{d\vec{r}(f(t))}{dt} \cdot f'(t)$$

Finally a very important rule about VVF derivatives is that the derivative VVF is orthogonal to the original VVF at every point (if the VVF is of constant magnitude), since:

$$\vec{r}(t) \cdot \vec{r}(t) = \left\|\vec{r}(t)\right\|^2$$

$$\frac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)] = \frac{d}{dt}[\left\|\vec{r}(t)\right\|^2]$$

$$\frac{d\vec{r}(t)}{dt} \cdot \vec{r}(t) + \vec{r}(t) \cdot \frac{d\vec{r}(t)}{dt} = 0$$

$$2\vec{r}(t) \cdot \frac{d\vec{r}(t)}{dt} = 0$$

$$\vec{r}(t) \cdot \frac{d\vec{r}(t)}{dt} = 0$$

Remember that if two vector have a dot product of 0 they are orthogonal.

Tangent Vectors

Given that we know the vector $\vec{r}(t_0)$ and $\vec{r}\,'(t_0)$ are perpandicular, the derivative vector is tangent to the curve traced out by $\vec{r}(t)$ at $t = t_0$. We define the normalized version of this derivative vector as the **tangent vector** ($\vec{T}(t)$):

Definition 3.2.2 Tangent Vectors

The **principal unit tangent vector** of $\vec{r}(t)$ is given by:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$$

Finally, we finish this section with integration of VVFs, which works how you would expect:

Definition 3.2.3 VVF Integral

Let f, g, and h be differentiable functions of t. Then:

$$\int \vec{r}(t)dt = \begin{bmatrix} \int f(t)dt \\ \int g(t)dt \\ \int h(t)dt \end{bmatrix}$$

3.3 Arc Length and Curvature

Recall that informally, arc length is the length of the curve between two points. We can compute the arc length using the following formula:

Definition 3.3.1 Arc Length

Let f, g, and h be differentiable functions of t. Then arc length (s) of a VV curve C between t values of a and b is:

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt = \int_a^b \left\|\vec{r}\,'(t)\right\| dt$$

We can then consider what happens if we make the upper limit of this value a function of t:

$$s(t) = \int_a^t \left\|\vec{r}\,'(u)\right\| du$$

This would result in s being a function of t. This is a useful function to have, as it returns the arc length of the curve from some starting value $t = a$ to the t value given as input to the function.

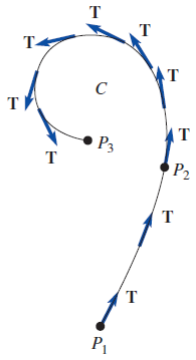
We can take this a step further and rearrange the function $s(t)$ for $t = t(s)$, and then replacing t with $t(s)$ in our function definition:

$$\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} \implies \vec{r}(s) = \begin{bmatrix} f(s) \\ g(s) \\ h(s) \end{bmatrix}$$

This is called the **arc-length parametrization** of the curve C. This is useful because now the parameter has more meaning, $\vec{r}(5)$ now returns the point on the curve which is 5 units along the curve away from $t = a$.

Curvature

If you look at the following graph, you can see the tangent vectors change as you move along the curve. The tangent vectors change a lot when the curvature is high, and the tangent vectors don't change much when the curvature is low. This is how we **define** curvature, as how fast the tangent vectors change:



Definition 3.3.2 Curvature

Let $\vec{r}(s)$ be the arc length parameterization of the curve C. The curvature κ at s is:

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

There are some alternative formulas for curvature:

$$\kappa = \frac{\left\| \vec{T}\,'(t) \right\|}{\left\| \vec{r}\,'(t) \right\|}$$

For a $3D$ curve:

$$\kappa = \frac{\left\| \vec{r}\,'(t) \times \vec{r}\,''(t) \right\|}{\left\| \vec{r}\,'(t) \right\|^3}$$

For a function $y = f(x)$

$$\kappa = \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$$

We now define the normal and binormal vectors of the VVF, similar to how we defined the tangent vector:

Definition 3.3.3 Normal and Binormal Vectors

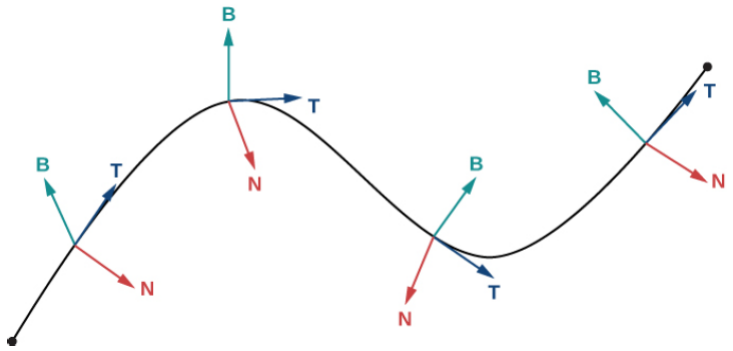
Let C be a three-dimensional **smooth** curve represented by $\vec{r}(t)$ over an open interval I. The principal unit normal vector is defined to be:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\left\|\vec{T}\,'(t)\right\|}$$

The binormal vector at t is defined as:

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

At every point $\vec{r}(t_0)$ on the curve C, the tangent, normal, and binormal vectors form a right handed system centred at $\vec{r}(t_0)$.



Another good technique to know is given some implicitly defined graph $(G(x, y) = c)$, you can determine the tangent and normal to the curve by using the following technique:

$$\vec{T} = \begin{bmatrix} 1 \\ y'(x_0) \end{bmatrix}$$

$$\vec{N} = \begin{bmatrix} -y'(x_0) \\ 1 \end{bmatrix}$$

Chapter 4: Differentiation of Functions of Several Variables

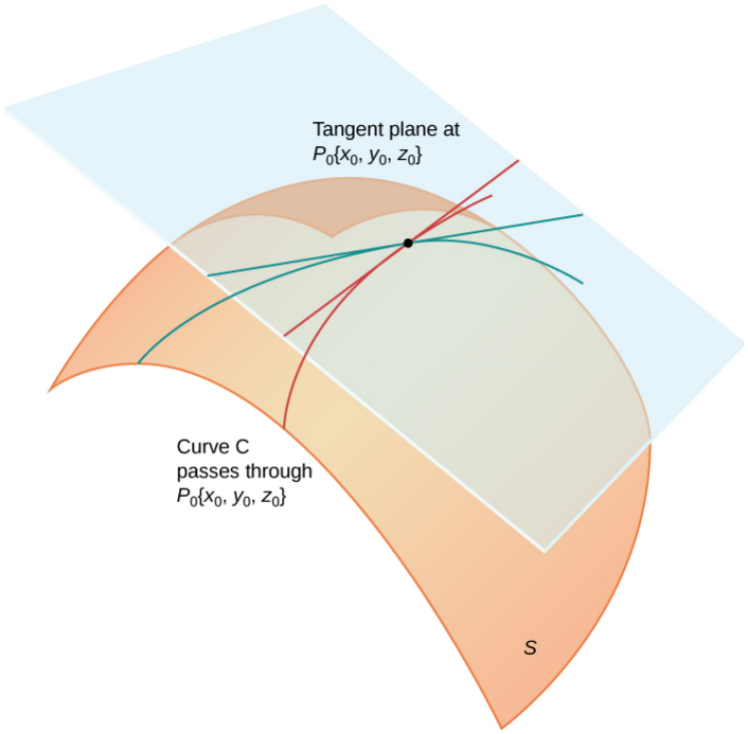
4.4 Tangent Planes and Linear Approximations

The tangent plane to a curve at a point P_0 is the plane which contains all the tangent lines to the curve at that point. We can obtain the equation of the tangent plane by the following formula:

Definition 4.4.1 Tangent Planes

Let S be the surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then the equation of the tangent plane to S at P_0 is given by:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$



4.6 Directional Derivatives and the Gradient

The gradient is a mathematical tool used to represent more complex ideas. The gradient is an operator which you can apply to a function, and returns a vector in which each component is the partial derivative of that function with respect to that component, as in:

Definition 4.6.1 Gradient

Given a function $f(x_1, x_2, \ldots, x_n)$ such that f_{x_1}, f_{x_2}, $\ldots f_{x_n}$ exist, the gradient of f, denoted:

$$\nabla f(x_1, x_2, \ldots, x_n) = \text{grad } f$$

... is defined as:

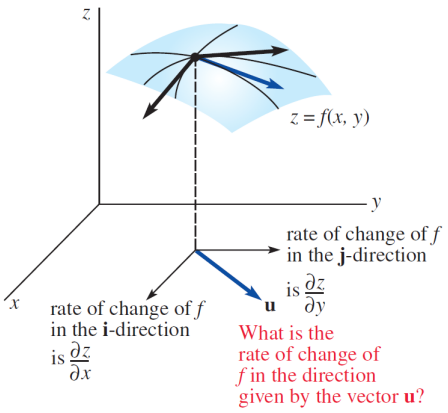
$$\nabla f(x_1, x_2, \ldots, x_n) = \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{bmatrix}$$

The gradient vector at any point on the surface is always normal to the tangent plane at that point.

Directional Derivatives

The partial derivative with respect to x can be interpreted as the rate of change of the function after taking one step in the $\hat{i}$ direction. Similarly, the partial derivative with respect to y can be interpreted as the rate of change of the function after taking one step in the $\hat{j}$ direction.

But what if we want to know the rate of change of the function after taking a step in the direction of an arbitrary unit vector $\hat{u}$? This is called the **directional derivative**.



Definition 4.6.2 Directional Derivative

The directional derivative of $z = f(x, y)$ in the directional of a unit vector defined as:

$$\hat{u} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

... is defined as:

$$D_{\hat{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + h\cos(\theta), y + h\sin(\theta)) - f(x, y)}{h}$$

Notice that this is really generalizing partial differentiation, as if you set $\theta = 0$ you get the definition of $\frac{\partial z}{\partial x}$, and if you set $\theta = \frac{\pi}{2}$ you get the definition of $\frac{\partial z}{\partial y}$.

There is however an easier way to compute the directional derivative:

Theorem 4.6.1 Computing the Directional Derivative

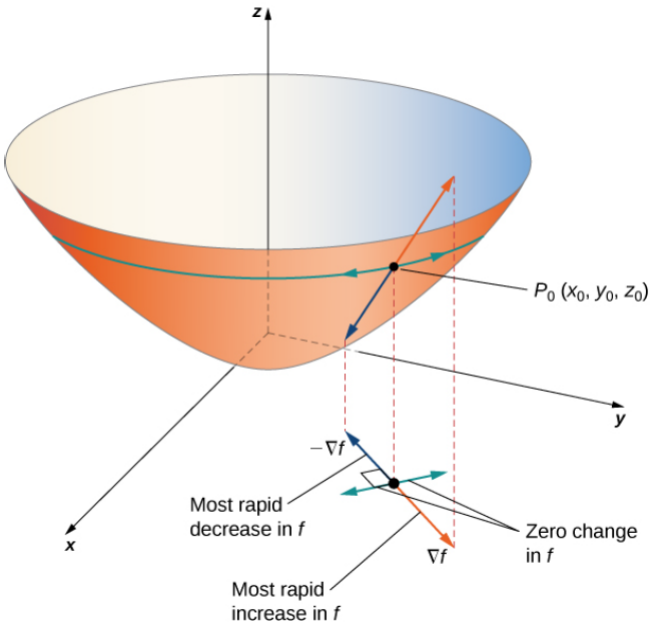
Let $f(x, y, z)$ be a differentiable fnction of three variables, and let $\hat{u}$ be a unit vector. Then the directioanl derivative of f in the direction of $\hat{u}$ is:

$$D_{\hat{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \hat{u}$$

Properties

The following is a list of some notable properties of the gradient and the directional derivative:

- If $\nabla f(x_0, y_0) = 0$, then $D_{\hat{u}} f(x_0, y_0) = 0$ for any $\hat{u}$.
- If $\nabla f(x_0, y_0) \neq 0$, then $D_{\hat{u}}(x_0, y_0)$ is maximized when $\hat{u}$ points in the same direction as $\nabla f(x_0, y_0)$, and the maximum value is $\left\|\nabla f(x_0, y_0)\right\|$.
- If $\nabla f(x_0, y_0) \neq 0$, then $D_{\hat{u}}(x_0, y_0)$ is minimized when $\hat{u}$ points in the opposite direction as $\nabla f(x_0, y_0)$, and the minimum value is $-\left\|\nabla f(x_0, y_0)\right\|$.



Chapter 5: Multiple Integration

In this section we study what integration means for a multi variable function.

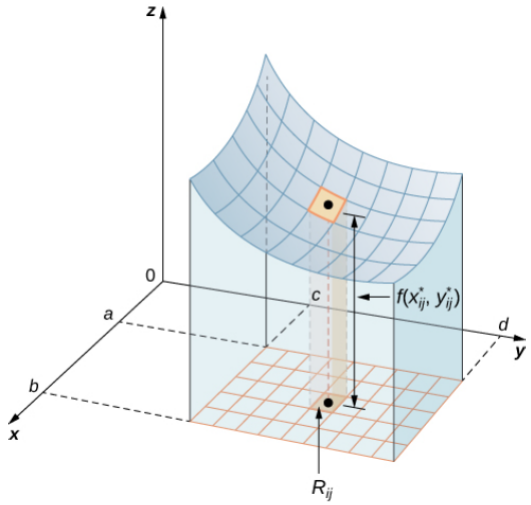
5.1 Double Integral over Rectangular Coordinates

In single-variable integration, we took the value of the function for every value between a and b, added them all together and called it:

$$\int_a^b f(x)$$

You can think of this as integrating on the line $y = 0$. We will study how to integrate on other lines, but first we will talk about integrating over areas instead.

In single variable functions, we divided the interval a to b into n subregions, evaluated the function at an arbitrary point in the subregion, and then added all those values together. We do a similar thing with integration of 2 variable functions but this time it is into regions R_k, as in:



The area of each of those squares is $\Delta x \cdot \Delta y = \Delta A$. The **double Riemann sum** definition of the double integral is as follows:

Definition 5.1.1 Double Integral

The **double integral** of the function $f(x, y)$ over the rectangular region R in the $xy-$plane is defined as:

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Notice that we do not put the definite bounds on the integral even though we are defining a definite integral, this is because we are not evaluating this function over an interval, we are evaluating this function for every point in a $2D$ space called R on the input plane.

The following is a list of important properties of the double integral. Assume that $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R, S and T are subregions of R, and that m and M are real numbers:

- The sum $f(x, y) + g(x, y)$ is integrable as:

$$\iint_R f(x,y) + g(x,y)dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$$

- If c is a constant, then $cf(x, y)$ is integrable as:

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$$

- If $R = S \cup T$ and $S \cap T = \varnothing$ except an overlap on the boundaries, then:

$$\iint_R = f(x,y)dA = \iint_S f(x,y)dA + \iint_T f(x,y)dA$$

- If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then:

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$$

- If $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, then:

$$m \cdot (\text{Area of R}) \leq \iint_R f(x,y)dA \leq M \cdot (\text{Area of R})$$

- If $f(x,y)$ can be factored into a product $g(x)h(y)$, then the definite double integral over the rectangular region $R = [a,b] \times [c,d]$ is:

$$\iint_R f(x,y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Evaluating Double Integrals over a Rectangular Region

Over a rectangular, it is pretty straightforward to evaluate using the following theorem:

Theorem 5.1.1 Fubini's Theorem

Suppose that $f(x,y)$ is continuous over a rectangular region $R = [a,b] \times [c,d]$. Then then the double integral is equal to either of the following:

$$\iint_R f(x,y)dA = \iint_R f(x,y)dx\ dy = \int_c^b \int_a^b f(x,y)dx\ dy$$

$$\iint_R f(x,y)dA = \iint_R f(x,y)dy\ dx = \int_a^b \int_c^d f(x,y)dy\ dx$$

This is illustrated by the following image:

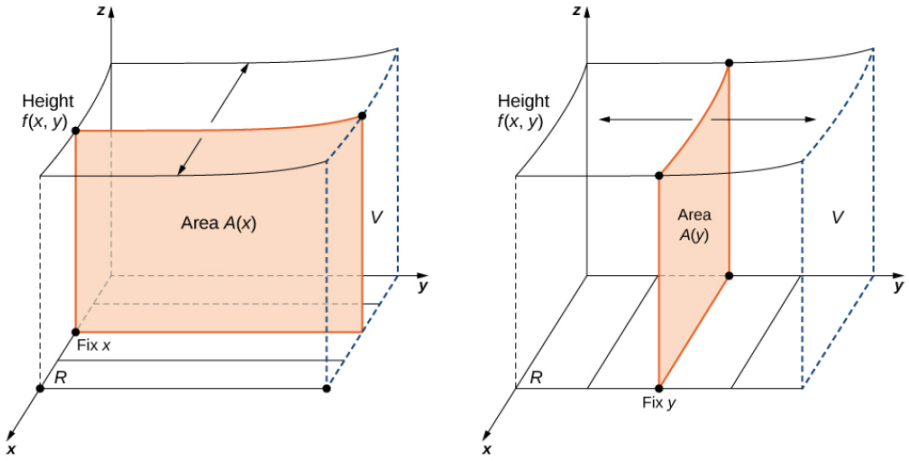


Figure 5.7 (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

So far we only know to evaluate double integrals over rectangular regions in the input space. In the next section we will learn how to evaluate them over more complex regions.

5.2 Double Integral over General Regions

Before we get to completely general regions of the input plane to integrate over, we study two simpler cases called *Type I* and *Type II* regions:

Definition 5.1.3 Type I and Type II Regions

A region D in the plane is of **Type I** if it lies between two vertical lines, and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$, that is:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

A region D in the plane is of **Type II** if it lies between two horizontal lines, and the graphs of two continuous functions $h_1(x)$ and $h_2(x)$, that is:

$$D = \{(x, y) \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

Examples of each are as follows:

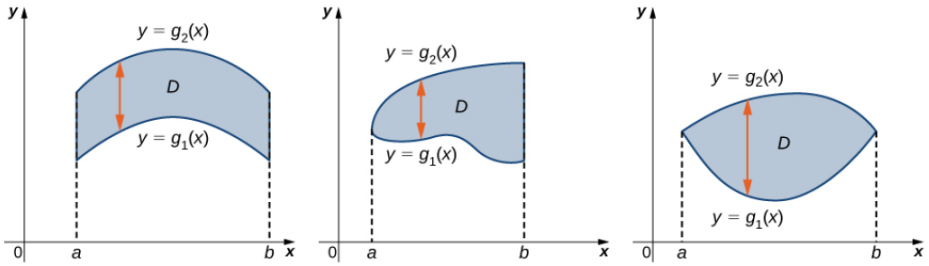


Figure 5.13 A Type I region lies between two vertical lines and the graphs of two functions of x.

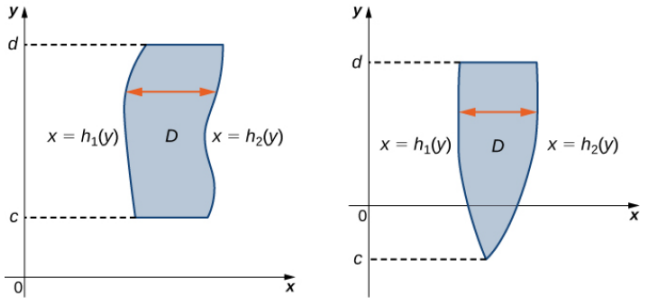


Figure 5.14 A Type II region lies between two horizontal lines and the graphs of two functions of y.

If we are integrating over either *Type I* or *II* regions then we have a direct formula we can use:

Theorem 5.2.1 Fubini's Theorem (Strong Form)

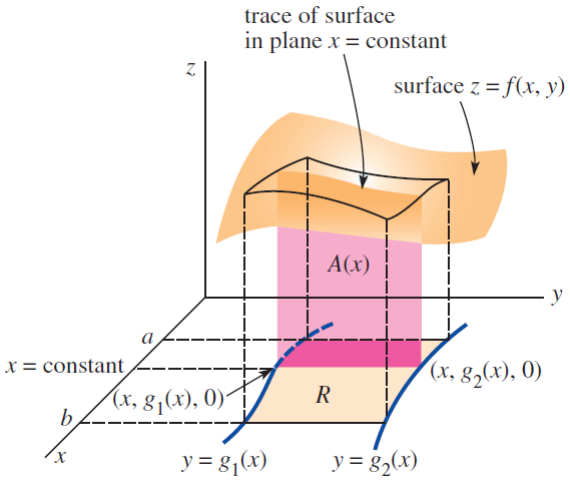
For a function $f(x,y)$ that is continuous on a region D of **Type I**, then:

$$\iint_D f(x,y)dA = \iint_D f(x,y)dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

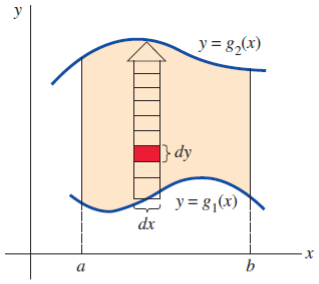
For a function $f(x,y)$ that is continuous on a region D of **Type II**, then:

$$\iint_D f(x,y)dA = \iint_D f(x,y)dx\,dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dx\,dy$$

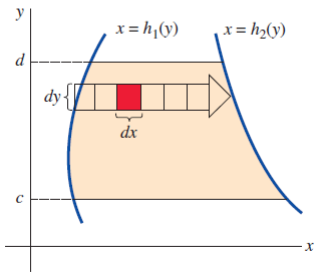
The idea of this theorem is to call the inner integral a slice of the volume called $A(x)$ and integrate that between the two constant bounds. The geometric interpretation for a *Type I* region is:



... additionally, it is effective to imagine the first integral integrating the function across a line in the x or y direction (depending on type I or type II) and then integrating that line between the constant bounds, as in:



(a) Region of Type I



(b) Region of Type II

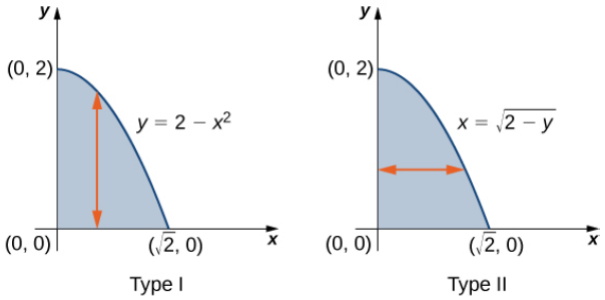
From here, if you have a particularly complex region to integrate over, you just need to decompose it into regions you know how to integrate over and use the following theorem to add them back together:

Theorem 5.2.2 Decomposing Regions

Suppose a region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at boundaries, then:

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

Finally, you can reverse the order of integration by changing the way you think about the region. A *Type I* region is *Type II* if you switch the independent and dependent variables around, for example:



Which shows us that:

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2}\,dy\,dx = \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\,dx\,dy$$

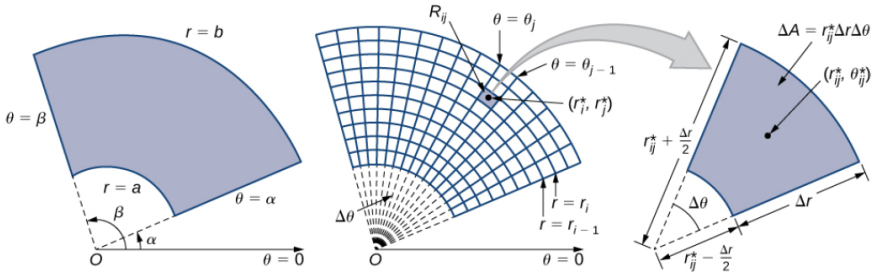
5.3 Double Integral in Polar Coordinates

Often it is easier to evaluate a double integral in polar coordinated. We want to be able to convert a rectangular double integral to a polar double integral. First we must define polar double integration.

In rectangular double integrals, we subdivided the region of the domain into subregions which were all rectangles, here we will instead need to use **polar rectangles** which are produced by varying θ from α to β, and r from a to b, as in:

$$R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

We then subdivide, find the area of each smaller region, evaluate the function at an arbitrary point in the region:



Definition 5.3.1 Polar Double Integral

The double integral of the function $f(r,\theta)$ over the polar rectangular region R in the $r\theta-$plane is defined as:

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r^*_{ij},\theta^*_{ij})\Delta A$$

Note also that:

$$\Delta A = r^*_{ij}\Delta r\Delta\theta$$

We can evaluate the polar integral over **polar rectangular** regions similar to how we do it for rectangular regions... given some polar rectangular region R:

$$R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

... then:

$$\iint_R f(r,\theta)dA = \iint_R f(r,\theta)r\ dr\ d\theta = \int_\alpha^\beta \int_a^b f(r,\theta)r\ dr\ d\theta$$

To convert a function to a polar double integral from rectangular, we use the following formula:

$$\iint_R f(x,y)dA = \iint_R f(r\cos(\theta), r\sin(\theta))r\ dr\ d\theta$$

This is achieved by making the following substitutions:

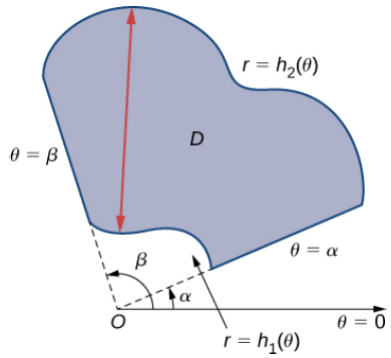
$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$dA = r\ dr\ d\theta$$

Other Types of Polar Regions

Now we want to be able to integrate over a more complex polar region. Similar to *Type I* and *Type II* in rectangular coordinates, in polar we have one type of region which is simple to evaluate over, in the form of:



Which can be written as:

$$D = \{(r, \theta) \mid h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$$

Then:

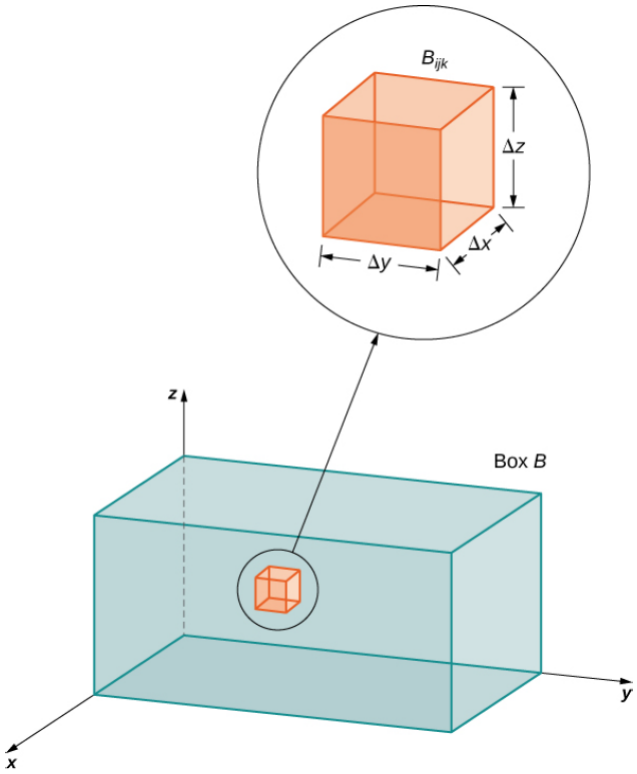
Theorem 5.3.1 Polar Double Integrals

If $f(r, \theta)$ is continuous on a general polar region D as described above, then:

$$\iint_D f(r,\theta) r \; dr \; d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta) r \; dr \; d\theta$$

5.4 Triple Integrals

In this section we define the triple integral, which is how you integrate functions of three variables. Visually you can imagine only the function's input space being some volume in $\mathbb{R}^3$. From there, we want to divide this region into sub-boxes with dimensions $\Delta x \times \Delta y \times \Delta z$.



Through the same process as before, of making sub-regions, taking an arbitrary point in each subregion to evaluate the function at, adding all those function values together and then taking the limit as the size of those subregions goes to zero, we get the formal definition of the triple integral:

Definition 5.4.1 Triple Integral

The **triple integral** of the function $f(x, y, z)$ over the rectangular box B in the $xyz-$space is defined as:

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ij}^*, y_{ij}^*, z_{ij}^*)\Delta x\Delta y\Delta z$$

All the properties of the double integral also apply to the triple integral, so they can be assumed.

You can imagine a triple integral begin what you get when you evaluate a function at every point inside a region of $\mathbb{R}^3$.

We begin the study by learning how to evaluate triple integrals over a rectangular box B given by:

$$B = [a,b] \times [c,d] \times [e,f] = \{(x,y,z) \in \mathbb{R}^3 \mid a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$$

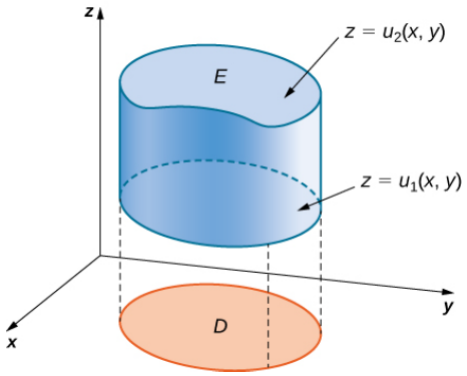
Theorem 5.4.1 Fibini's Theorem for Triple Integrals

If $f(x,y,z)$ is continuous on a rectangular box B as described above, then:

$$\iiint_B f(x,y,z)dV = \int_e^f \int_c^d \int_a^b f(x,y,z)dx\ dy\ dz$$

... or any of the 5 other permutations of $dx\ dy\ dz$.

Now we want evaluate a triple integral over more complex volumes, we begin by looking at a volume which is between two curves, over some region D on the $xy-$plane, as in:



We can think of D as the projection of the volume onto the $xy-$plane. We then use the following theorem to evaluate a triple integral over this kind of space.

Theorem 5.4.2 Triple Integrals

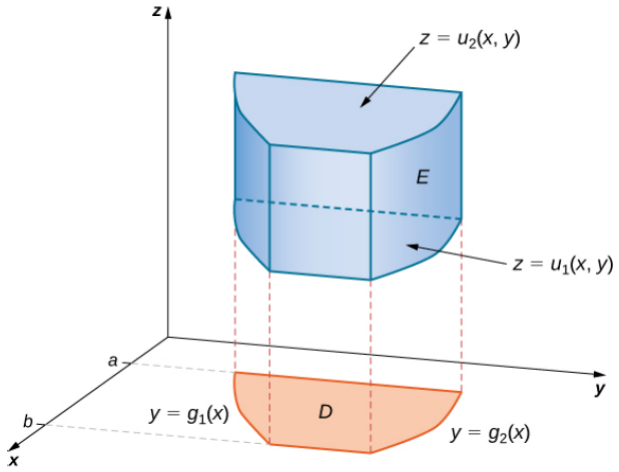
The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region E whch is overtop some region D on the $xy-$plane between two surfaces $u_1(x, y)$ and $u_2(x, y)$, as in:

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

... is:

$$\iiint_E f(x, y, z)dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dxdA$$

If that region D is of *Type I* as in:

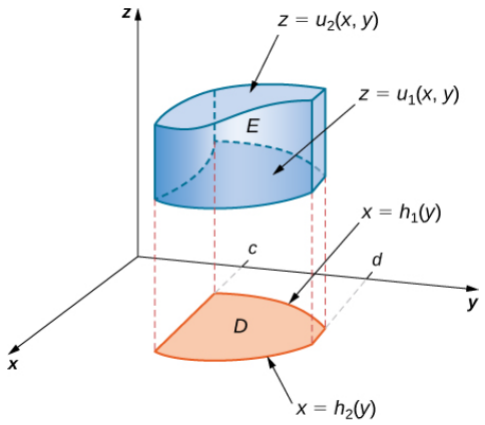


$$E = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$

Then the integral becomes:

$$\iiint_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\ dy\ dx$$

If that region D is of *Type II* as in:



$$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$$

Then the integral becomes:

$$\iiint_E f(x, y, z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\ dx\ dy$$

Recall that there is always two ways to write the double integral, and so we can change the variables as well in a triple integral in the same way.

5.5 Triple Integrals in Cylindrical and Spherical Coordinates

In this section we define how we compute triple integrals in cylindrical (r, θ, z) and spherical (r, θ, ϕ) coordinates.

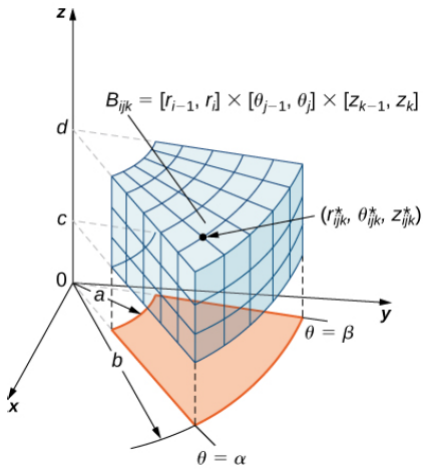
Cylindrical Triple Integrals

Integrating in cylindrical coordinates is usefor for solids that are symmetric around an axis, such as cylinders or cones.

If we define some box B in cylindrical coordinates:

$$B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}$$

Then we want to divide this box into sub-boxes, evaluate the function in each box, sum them all together and then take the limit as the size of the boxes goes to zero.



Definition 5.5.1 Cylindrical Triple Integral

Considering a cylindrical box B as described above, if the function $f(r, \theta, z)$ on B then we can define the triple integral in cylindrical coordinates as the limit of a triple Riemann sum:

$$\iiint_B f(r, \theta, z) dV =$$

$$\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z$$

It is worth noting:

$$\iiint_B g(x, y, z) dV = \iiint_B g(r\cos(\theta), r\sin(\theta), z) r \; dr \; d\theta \; dz$$

The following theorem is as you'd expect for integration of cylindrical coordinates:

Theorem 5.5.1 Cylindrical Triple Integrals

Suppose that $g(x, y, z)$ is continuous on a rectangular box B as described by:

$$B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}$$

... then:

$$\iiint_B g(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b g(r\cos(\theta), r\sin(\theta), z)r\, dr\, d\theta\, dz$$

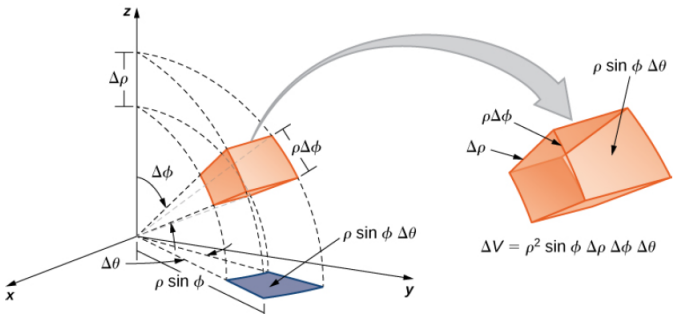
Spherical Triple Integrals

Integrating in cylindrical coordinates is use for for solids that are symmetric a point, such as spheres or cones.

If we define some box B in spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \lambda\}$$

Then we want to divide this box into sub-boxes, evaluate the function in each box, sum them all together and then take the limit as the size of the boxes goes to zero.



Definition 5.5.2 Spherical Triple Integral

Considering a spherical box B as described above, if the function $f(\rho, \theta, \phi)$ on B then we can define the triple integral in spherical coordinates as the limit of a triple Riemann sum:

$$\iiint_B f(\rho, \theta, \phi) dV =$$

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \phi^*_{ijk})(\rho^*_{ijk})^2 \sin(\phi) \Delta\rho \Delta\theta \Delta\phi$$

It is worth noting that:

$$dV = \rho^2 \sin(\phi) d\rho \; d\theta \; d\phi$$

The following theorem is as you'd expect for integration of spherical coordinates:

Theorem 5.5.2 Spherical Triple Integrals

Suppose that $g(\rho, \theta, \phi)$ is continuous on a rectangular box B as described by:

$$B = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \lambda\}$$

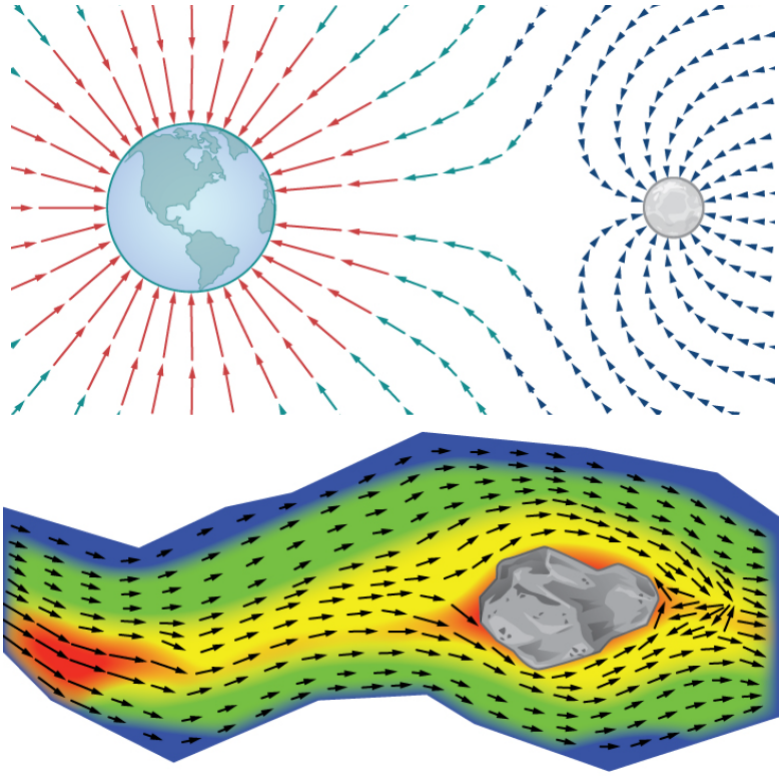
... then:

$$\iiint_B g(\rho, \theta, \phi) dV = \int_\gamma^\lambda \int_\alpha^\beta \int_a^b g(\rho, \theta, \phi) \rho^2 \sin(\phi) d\rho \; d\phi \; d\theta$$

Chapter 6: Vector Calculus

6.1 Vector Fields

A vector field is a way to assign every point in space with a vector. The following are some examples:



Definition 6.1.1 Vector Fields

A vector field $\vec{F}$ in $\mathbb{R}^2$ is an assignment of a 2−dimensional vector $\vec{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. You can think of a vector field in $\mathbb{R}^2$ as:

$$\vec{F} \quad \mathbb{R}^2 \to \mathbb{R}^2$$

... as in a function with two inputs and two outputs.
A vector field $\vec{F}$ in $\mathbb{R}^3$ is an assignment of a 3−dimensional vector $\vec{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. You can think of a vector field in $\mathbb{R}^3$ as:

$$\vec{F} \quad \mathbb{R}^3 \to \mathbb{R}^3$$

... as in a function with three inputs and three outputs.

Typically, we write them as vectors:

$$\vec{F}(x,y,z) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$$

... additionally given some vector field, we can normalize all its components to get a unit vector field, as in:

$$\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \rightarrow \vec{U} = \frac{1}{\left\|\vec{F}\right\|} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

We now discuss gradient fields, which are a special type of vector field, also called conservative fields. Recall the gradient of a function is a vector assigned to every point in it's domain, the gradient is a vector field. If a vector field is the gradient field of some function f, then it is a gradient field.

Definition 6.1.2 Gradient Fields

A vector field $\vec{F}$ is a gradient field if there exists some scalar function f such that $\nabla f = \vec{F}$. We call f a **potential function** for $\vec{F}$.

This should remind you of integration, if you "indefinitely integrate" the vector field, you get a potential function whose gradient field is the vector field in question. But if potential functions differ by a constant, they would have the same gradient field, which brings us to the next theorem:

Theorem 6.1.1 Potential Functions

Let $\vec{F}$ be a conservative vector field on an open and connected domain. Let f and g be functions such that:

$$\nabla f = \vec{F}$$

$$\nabla g = \vec{F}$$

... then there is a constant C such that:

$$f = g + C$$

Given some arbitrary vector field, it is more likely not conservative. This is because a conservative vector field must pass the following test. It should be noted that the following test can only be used to prove that a vector field is **not** conservative, and not that it **is**.

Theorem 6.1.2 Cross-Partial Property

Let $\vec{F}$ be a vector field with components who have continuous second-order mixed-partial derivatives on its domain.
If $\vec{F} = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$ is conservative then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

If $\vec{F} = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$ is conservative then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

6.2 Line Integrals

We would now like to learn how to integrate in a vector field. There are multiple different kinds of integrals of vector fields. In this section we begin discussing the *line integral*. First we learn what it means to take the line integral of a regular function (called a *scalar line integral*) and then we learn how to take the line integral of a vector field (called a *vector line integral*).

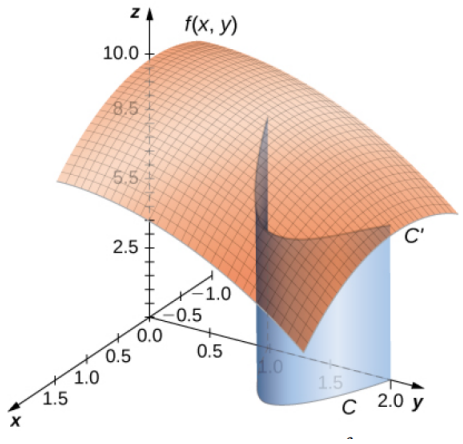
Scalar Line Integrals

The notation:

$$\int_a^b f(x)dx$$

... means we want to integrate the function along the $x - axis$ from a to b.

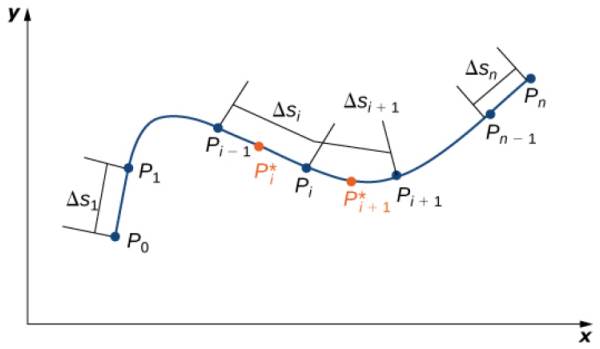
If we have a function of 2 variables, you can imagine drawing any arbitrary curve on the domain plane, call this curve C. We know how to integrate all the values of a function on the $2D$ input plane, but what if we just want to integrate the function along the curve C, this can be visualized as below:



... and we denote this:

$$\int_C f(x,y)\ ds$$

We construct the line integral by breaking up the curve into a bunch of straight line segments which have constant Δt.



Note that in the picture $P_i = \vec{r}(t_i)$ You can then approximate the line integral by adding up rectangles with height $f(\vec{r}(t_0))$ and width Δs.

Definition 6.2.1 Scalar Line Integral

Let f be a function whose domain contains a curve C parametrized by the function $\vec{r}(t)$, $a \leq t \leq b$. The scalar line integral of f along C is defined as:

$$\int_C f ds = \lim_{n\to\infty} f(\vec{r}(t^*))\Delta s_i$$

We can simplify this process by changing the integration variable to t, by recalling (for a 3-variable function):

$$\frac{ds}{dt} = \left\|\vec{r}\,'(t)\right\| = \sqrt{(x'(t))^2 + y'(t))^2 + z'(t))^2}$$

... meaning:

$$ds = \sqrt{(x'(t))^2 + y'(t))^2 + z'(t))^2}dt$$

... which brings us to the next theorem:

Theorem 6.2.1 Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes C parameterized by given by (in $3D$):

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

... $a \leq t \leq b$. Then:

$$\int_c f(x,y,z)ds = \int_a^b f(\vec{r}(t))\sqrt{(x'(t))^2 + y'(t))^2 + z'(t))^2}dt$$

Note that parameterization of the curve C has no effect on the line integral as long as the curve is traversed once as $a \leq t \leq b$.

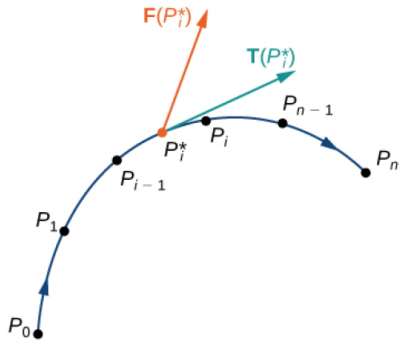
Additionally, we can apply this idea to define arc length of a curve C to be just the integral of 1 as t varies, as in:

$$\text{Arc Length of } C = \int_C ds$$

Vector Line Integrals

We now want to integrate the vector field along some curve C. The meaning of this is we want a way to measure how much the vector field is aligned with the curve, and if the vector field represents a force field, how much work the force field is doing on an object along that path (Recall work $= \vec{F} \cdot \vec{d}$).

Imagine taking a curve in a vector field. At every point on the curve there would be a corresponding vector from the vector field, and also a tangent vector of the curve, as in:



If we take the dot product of those two vectors, we get some kind of measure of how aligned those two vectors are. If we do this for ever point along the arclength of the curve, we get the following definition of the vector line integral:

Definition 6.2.2 Vector Line Integral

The vector line integral of a vector field $\vec{F}$ along a curve C parameterized by $\vec{r}(t)$ is:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \lim_{n\to\infty} \vec{F}(\vec{r}(t^*)) \cdot \vec{T}(\vec{r}(t^*))\Delta s_i$$

We want a way to evaluate these vector line integrals, and so we want to express it in terms of t. Recall that:

$$\vec{T} = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$$

$$ds = \left\|\vec{r}\,'(t)\right\| dt$$

We can combine these facts into:

$$\vec{F} \cdot \vec{T} ds = \vec{F} \cdot \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} \left\|\vec{r}\,'(t)\right\| dt = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt$$

... and thus:

$$\int_C \vec{F} \cdot \vec{T} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt = \int_C \vec{F} \cdot d\vec{r}$$

... note at the end we introduce a new notation which reflects the fact that to do the line integral you only need information about $\vec{F}$ and the derivative of $\vec{r}$:

$$d\vec{r} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

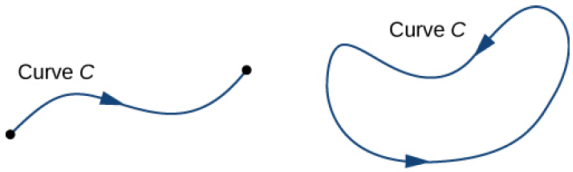
Another standard notation for line integrals, given that the vector field is in the form:

$$\vec{F} = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$$

... is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz = \int_a^b \left(P(\vec{r}(t)) \frac{dx}{dt} + Q(\vec{r}(t)) \frac{dy}{dt} + R(\vec{r}(t)) \frac{dz}{dt} \right) dt$$

With scalar line integrals, neither the parameterization nor direction of curve matters. With vector line integrals the orientation of the curve matters, as in if you go from left to right on a curve, you would get a different vector line integral compared to if you integrated the other way. This make sense, as the work done by gravity by moving up a fill is exactly negative the work done by gravity moving down a hill.



The opposite orientation of a curve C is denoted $-C$, and so:

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

The following is a list of properties the vector line integral has:

Theorem 6.2.2 Evaluating a Scalar Line Integral

Let $\vec{F}$ and $\vec{G}$ be continuous vector fields with domains that include the oriented smooth curve C. Then:

1. $$\int_C (\vec{F} + \vec{G}) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$$

2. $$\int_C k\vec{F} \cdot d\vec{r} = k \int_C \vec{F} \cdot d\vec{r}, \text{ where } k \in \mathbb{R}.$$

3. $$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot r\vec{r}$$

4. Suppose instead that C is a piece wise smooth curve in the domain of $\vec{F}$, where $C = C_1 + C_2 + \cdots + C_n$, then:
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \cdots + \int_{C_n} \vec{F} \cdot d\vec{r}$$

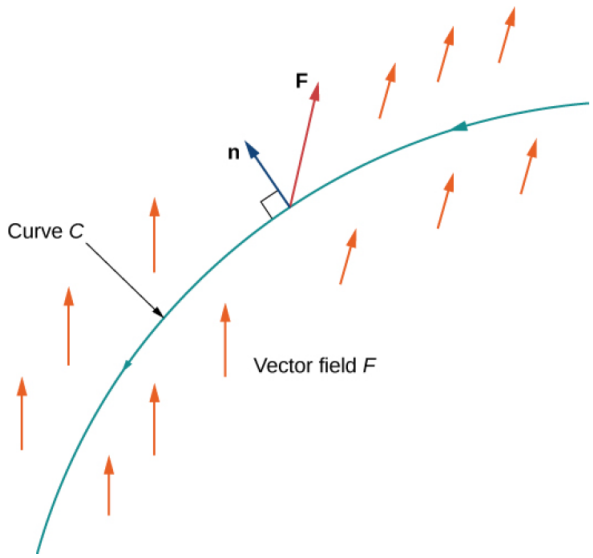
Flux and Circulation

Flux is the measure of flow of the vector field across some curve C. In a sense, we want to know for every vector on the line in the vector field, what component of that vector is normal to the curve at that point, which leads us to the definition of flux:

Definition 6.2.3 Flux

The flux of $\vec{F}$ across C is:

$$\int_C \vec{F} \cdot \frac{\vec{n}(t)}{\|\vec{n}(t)\|} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) dt$$



Note that for a curve parameterized by:

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

... the normal vector at any point is:

$$\vec{n}(t) = \begin{bmatrix} y'(t) \\ -x'(t) \end{bmatrix}$$

Circulation of $\vec{F}$ along a curve C measures the tendency of a fluid to move in the direction of the curve *where* C is a closed curve. The only difference is that C is a closed curve and so it forms a loop.

We use special notation for a circulation integral:

$$\oint_C \vec{F} \cdot \vec{T} ds$$

... which can then be evaluated as a regular line integral.

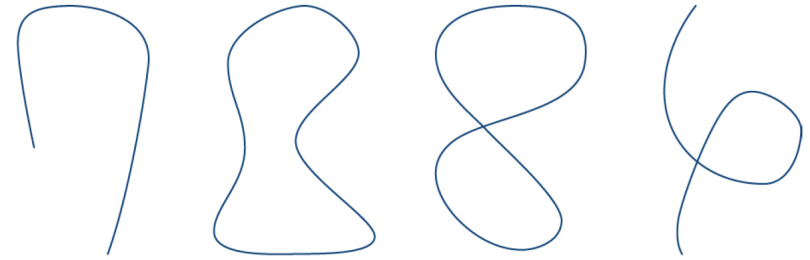
6.3 Conservative Vector Fields

In this section we continue our theoretical study of vector fields. Just like in single variable calculus, if a function is integrable on an interval, we have the fundamental theorem of calculus. There is a similar concept which can be applied to conservative vector fields, namely the *Fundamental Theorem of Line Integrals*.

Before we continue, we must define some concepts relating to curves and regions:

Definition 6.3.1 Curve Definitions

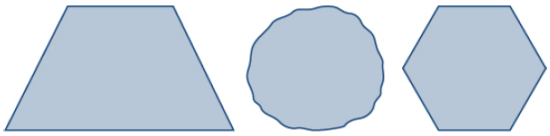
A curve C is a **closed curve** if there is a paramterization $\vec{r}(t), a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $\vec{r}(a) = \vec{r}(b)$.
A curve C is a **simple curve** if it does not cross itself. Meaning the parameterization is one-to-one on (a, b).



(a) Simple, not closed (b) Simple, closed (c) Not simple, closed (d) Not simple, not closed

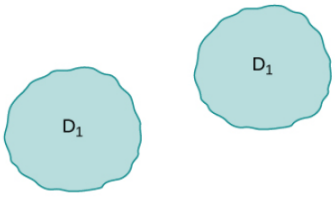
Definition 6.3.2 Region Definitions

A region D is a **connected region** if, for any two points, there is a path between them entirely inside D.
A region D is a **simply connected region** if any closed curve C entirely within D can be shrunken continuously to a point while staying entirely inside D. In two dimensions this means there are no holes.



(a) Simply connected regions

(b) Connected regions that are not simply connected



(c) A region that is not connected

If $\nabla f = \vec{F}$ is a gradient field, and we think of ∇ as a derivative, then f can be thought of as an anti-derivative of $\vec{F}$, also called the *potential function*. So just like in single variable integrals where we can evaluate the whole thing using only the value of the anti-derivative at the end points, if the anti-derivative exists, we can do the exact same thing with line integrals in conservative fields.

Theorem 6.3.1 Fundamental Theorem for Line Integrals

Let C be a piecewise smooth curve with parameterization $\vec{r}(t), a \leq t \leq b$. Let f be a function with first order partial derivative that exist and are continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof: We already know that:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) dt$$

... and since (using the chain rule):

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

... we can determine that:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \frac{d}{dt} f(\vec{r}(t))dt$$

... then by the fundamental theorem of calculus:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(t)) \mid_a^b$$

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

This lets us solve line integrals just like we solve single variable, by determining an anti derivative and then evaluating at the end points. One important note is that any continuous function has an antiderivative, **not every** continuous vector field has a potential function, only (by definition) gradient fields do. We will spend some time learning about how to test if a vector field is a gradient field. If a vector field is not a gradient field, you can evaluate the line integral how we did in the previous section.

There are two important consequences of the fundamental theorem for line integrals:

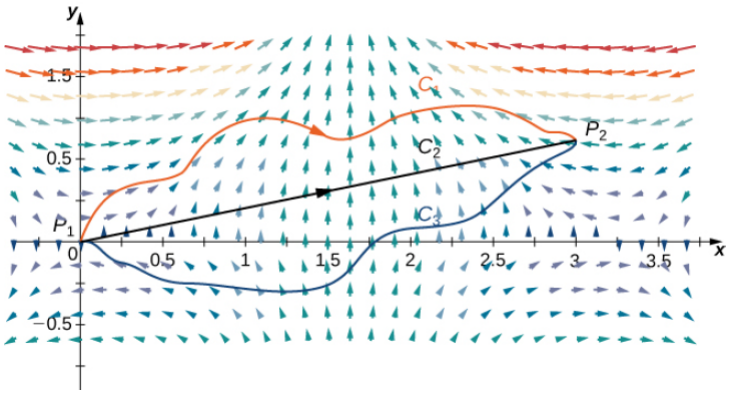
1. Circulation in a conservative field is 0.

   This is because in a circulation C is a closed curve (meaning $\vec{r}(a) = \vec{r}(b)$) and so $f(\vec{r}(a)) = f(\vec{r}(b))$, or in other words

   $$\oint_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

2. Line integrals of conservative fields are **independent of path**, meaning they only depend on the endpoints of the curve and not the path to get between them.

If $\vec{F}$ is conservative, then $\vec{F}$ is independent of path. The converse is sometimes true, if $\vec{F}$ is independent of path, and the domain D of $\vec{F}$ is open and connected, then $\vec{F}$ is conservative.



Finding a Potential Function

When using the fundamental theorem for line integrals, we want to know a potential function for the vector field. The following is a general strategy:

Strategy 6.3.1 Finding Potential Functions in $\mathbb{R}^2$

Given some vector field in the form:

$$\vec{F} = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

... you can find a potential function f such that $\nabla f = \vec{F}$ by:

1. Integrate $P(x,y)$ with respect to x. This will be a function in the form $g(x,y) + h(y)$.
2. Take the partial derivative of $g(x,y) + h(y)$ with respect to y, resulting in the function $g_y(x,y) + h'(y)$.
3. Make the equation:

$$g^y(x,y) + h'(y) = Q(x,y)$$

... to determine $h'(y)$.

4. Integrate $h'(y)$ with respect to y to find $h(y)$.
5. Any function in the form:

$$f(x,y) = g(x,y) + h(y) + C$$

... if a potential function for $\vec{F}$.

Strategy 6.3.2 Finding Potential Functions in $\mathbb{R}^3$

Given some vector field in the form:

$$\vec{F} = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$$

... you can find a potential function f such that $\nabla f = \vec{F}$ by:

1. Integrate $P(x,y,z)$ with respect to x. This will be a function in the form $f = g(x,y,z) + h(y,z)$.
2. Take the partial derivative of $f = g(x,y,z) + h(y,z)$ with respect to y, resulting in the function $f_y = g_y(x,y,z) + h_y(y,z)$.
3. Make the equation:

   $$g_y(x,y,z) + h_y(y,z) = Q(x,y,z)$$

   ... to determine $h_y(y,z)$. If at this point it is impossible for $h_y(y,z)$ to exist, then no potential function exists.
4. Integrate $h_y(y,z)$ with respect to y to determine $h(y,z)$, resulting in some $h(y,z) = q(y,z) + p(z)$.
5. Make the equation:

   $$g_z(x,y,z) + q_z(y,z) + p'(z) = R(x,y,z)$$

   ... to determine $p'(z)$.
6. Integrate $p'(z)$ with respect to z to determine $p(z)$.
7. Any function in the form:

   $$f(x,y,z) = g(x,y,z) + q(y,z) + p(z) + C$$

   ... if a potential function for $\vec{F}$. We now know $g(x,y,z)$ from step 1 and $q(y,z)$ from step 4, and $p(z)$ from step 6.

Test for Conservative Fields

We need to know if a given vector field is conservative, we can do this by:

Theorem 6.3.2 Test for Conservative Vector Fields

Let $\vec{F} =< P, Q, R >$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ thoughout D if and only if $\vec{F}$ is conservative.

6.4 Green's Theorem

We will finish this course by understanding three important theorems:

1. Green's Theorem (6.4)
2. Stokes' Theorem (6.7)
3. The Divergence Theorem (6.8)

The sections in between are added theory to work up to those final two theorems.

Green's theorem connects the circulation of a vector field, with the double integral of the area enclosed. This is a direct continuation of the idea that to evaluate an integral, we can use information about the boundary of integration. We will now formally state Green's theorem:

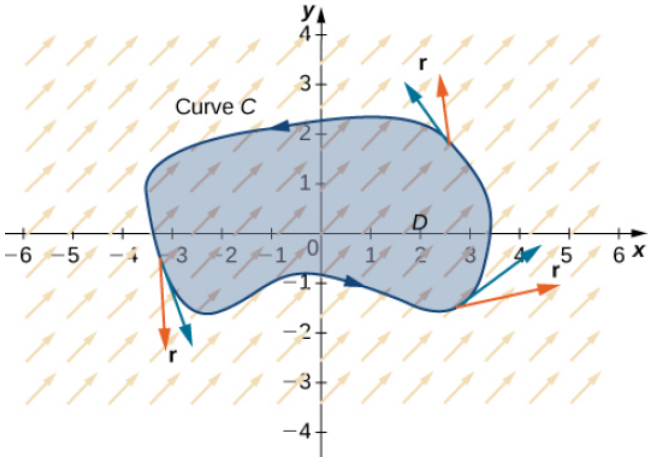
Theorem 6.4.1 Green's Theorem, Circulation Form

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise. Let $\vec{F}$ be a vector field in the form:

$$\vec{F} = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

... with continuous partial derivatives on D. Then:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA$$



There are many notes to make about this theorem:

1. Since:
$$\int_C P dx + Q dy = \int_C \vec{F} \cdot \vec{T} dt$$
... this is also called *tangential form* of Green's theorem.

2. Green's theorem only applies to vector fields in $\mathbb{R}^2$.

3. The first two expression are just different notations for the line integral, but the magic is that it is equal to the double integral on the right.

4. The theorem can be quickly proven for conservative vector fields since the circulation integral is equal to 0 because of the fundamental theorem of line integrals, and if $\vec{F}$ is conservative then $Q_x = P_y \implies Q_x - P_y = 0$.

5. The theorem can be used to turn a tricky line integral into a simpler double integral, or a tricky double integral into a simpler line integral.

6. To use this theorem to calculate a double integral, we can use a vector field with the property:
$$Q_x - P_y = 1$$
... such as:
$$\vec{F} = \begin{bmatrix} -\frac{y}{2} \\ \frac{x}{2} \end{bmatrix}$$
... and then use the following process:
$$\text{Area} = \iint_D dA$$
$$= \iint_D (Q_x - P_y) dA$$

$$= \int_C \vec{F} \cdot d\vec{R}$$

$$= \frac{1}{2} \int_C -ydx + +xdy$$

7. The previous theorem was labelled circulation form because there is another form of Green's theorem which will be discussed later. This other form is a direct result of the main Green's theorem (circulation form) and so really if you want to think of *the* Green's theorem, it would be this one.

We now discuss the *flux form* of Green's theorem:

Theorem 6.4.2 Green's Theorem, Flux Form

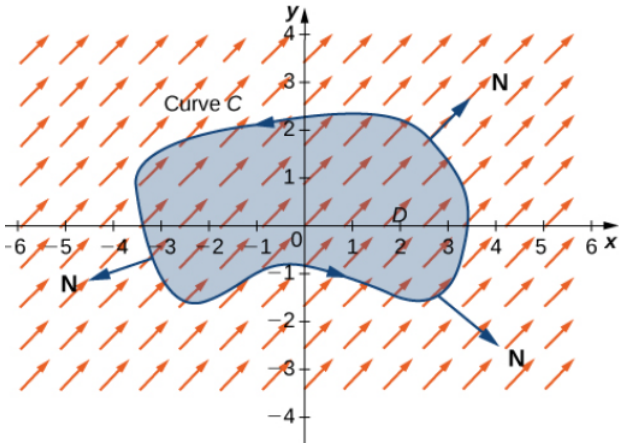
Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise. Let $\vec{F}$ be a vector field in the form:

$$\vec{F} = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

... with continuous partial derivatives on D. Then:

$$\oint_C \vec{F} \cdot \vec{N} ds = \iint_D (P_x + Q_y) dA$$

This extension of Green's theorem relates the flux through a closed curve C to the double integral of the bounded region.



This form of the theorem is useful in calculating the flux across a curve by turning it into a double integral.

Source-Free Vector Fields

A source free-vector field $\vec{F}$ is one where the flux of $\vec{F}$ around any closed curve is 0. Equivalently:

1. The flux $\oint \vec{F} \cdot \vec{N} ds$ across any closed curve C is zero.
2. Flux is independent of path.
3. There is a **stream function** $g(x, y)$ for $\vec{F}$. A **stream function** is like a potential function but for source-free fields. A stream function for $\vec{F} =< P, Q >$ is a function g such that:

$$P = g_y$$

$$Q = -g_x$$

... geometrically this means that the vector field is always tangential to the curve since:

$$\vec{F}(a, b) \cdot \nabla g(a, b) = 0$$

... for any point (a, b) in the domain of g.

4. $$P_x + Q_y = 0$$

6.5 Divergence and Curl

In this section we study two important operations which can be applied to a vector field, namely:

$$\text{Divergence: } \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$\text{Curl: } \operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

Divergence

Divergence is an operation which can be applied to a vector field and returns a scalar value. Divergence at a point measures the tendency of the vector field to flow out of the point. A positive divergence means more flow is coming out of the point than coming in. Effectively it is the flux per unit volume at the point. The following is its formal definition:

Definition 6.5.1 Divergence

If:

$$\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

... then the **divergence** of $\vec{F}$ is defined by:

$$\operatorname{div} \vec{F} = P_x + Q_y + R_z$$

This can be interpreted as the dot product between the gradient vector:

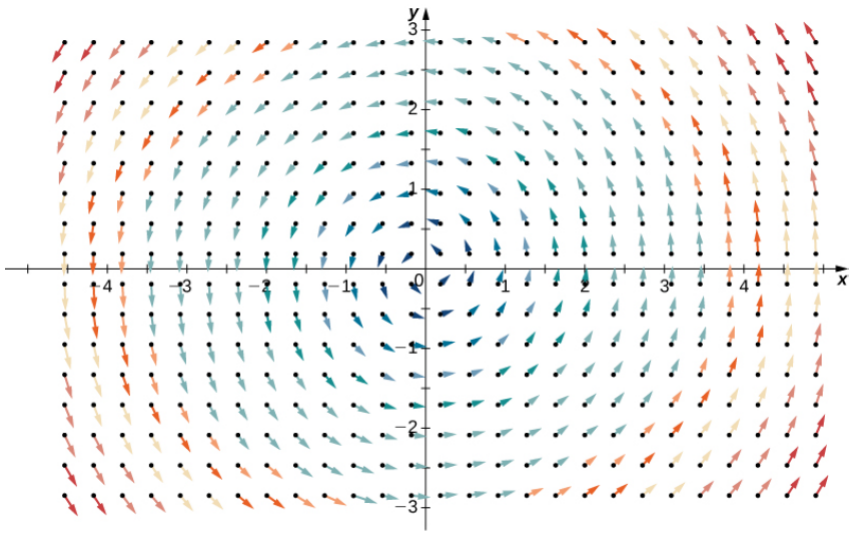
$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

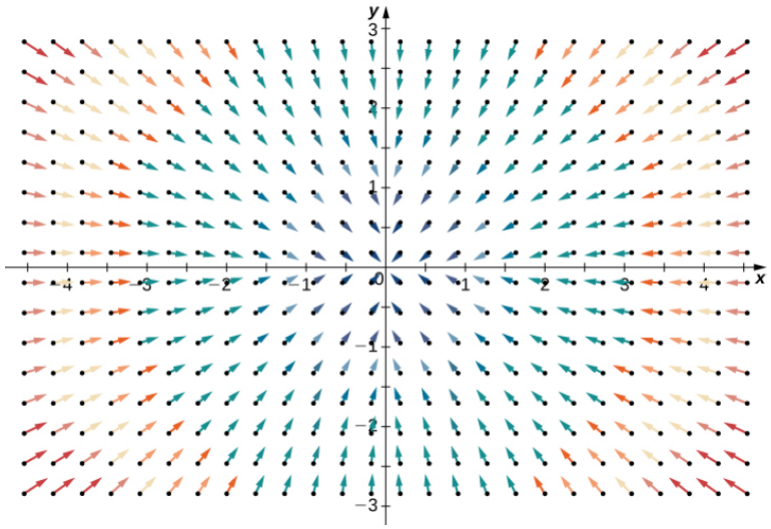
... and the vector field $\vec{F}$:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

Physically, divergence can be interpreted by imagining fluid flow. If the divergence at a point is positive, then the surroundings of the point act as a **source** of water. If the divergence at a point is negative, the surroundings act as a **sink** that sucks in water. If the divergence is equal to 0 then we say the fluid is incompressible.

The following image has zero divergence, and then negative divergence:





Divergence and Green's Theorem

Recall the flux form of Green's theorem:

$$\oint_C \vec{F} \cdot \vec{N} ds = \iint_D P_x + Q_y dA$$

Note that the divergence of a $2D$ vector field is:

$$\operatorname{div} \vec{F} = P_x + Q_y$$

... and so Green's theorem becomes:

$$\boxed{\oint_C \vec{F} \cdot \vec{N} ds = \iint_D \operatorname{div} \vec{F} dA}$$

If you think of the divergence as a type of derivative, then this continues the trend of being able to evaluate an integral (in this case a double integral) by using information about the anti-derivative at the boundary. Using divergence we can see that Green's theorem is directly a higher dimensional analogue to the fundamental theorem of Calculus.

Curl

The second operation we can apply to vector fields is *curl*. The curl of $\vec{F}$ at a point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector.

To visualize this, imagine putting a paddlewheel in the vector field, every point in the vector field can be assigned a vector with magnitude related to how much the fluid is swirling, and direction related to the axis around which it tends to swirl.

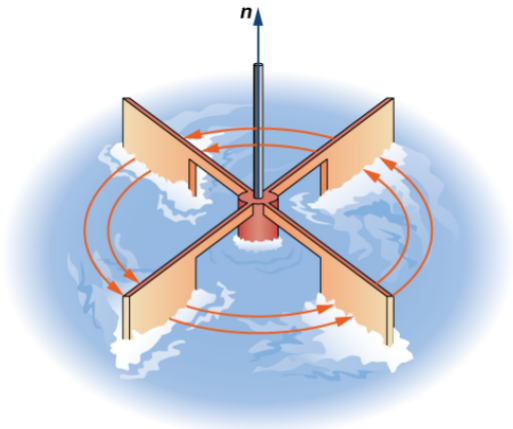


Figure 6.54 To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Definition 6.5.2 Curl

If:

$$\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

... is a vector field, then $\operatorname{curl} \vec{F}$ is defined as:

$$\operatorname{curl} \vec{F} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$$

... if $\vec{F}$ is in $\mathbb{R}^2$ then:

$$\operatorname{curl} \vec{F} = \begin{bmatrix} 0 \\ 0 \\ Q_x - P_y \end{bmatrix}$$

In general, computation of curl can be interpreted as the cross product of the gradient vector with the vector field:

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

The following is a useful theorem for conservative vector fields:

Theorem 6.5.1 Curl Test

Let $\vec{F} =< P, Q, R >$ be a vector field in space on a simply connected domain. If $\operatorname{curl} \vec{F} = 0$ then $\vec{F}$ is conservative.

Curl and Green's Theorem

Recall the circulation form of Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y dA$$

Note that for a vector field in $\mathbb{R}^2$:

$$\operatorname{curl} \vec{F} = (Q_x - P_y)\hat{k}$$

This implies:

$$\operatorname{curl} \vec{F} \cdot \hat{k} = Q_x - P_y$$

... and so Green's theorem can once again be rewritten as:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot \hat{k} dA}$$

If we interpret the curl of a vector field as a type of derivative, then once again this shows that you can evaluate an integral by knowing information about its anti-derivative around the boundary of integration.

6.6 Surface Integrals

A surface integral is the two dimensional analogue to the line integral. Let's remind ourselves about how we define our integrals:

- A regular integral is what you get when you evaluate a function at every x value along the $x - axis$ and add them all together.
- A scalar line integral is what you get when you evaluate a function at every point on any curve (regular integral is a special case of this) and add them all together.
- A double integral is what you get when you evaluate a function at every point inside a region of the $xy - plane$, and then add them all together.
- A **scalar surface integral** is what you get when you evaluate a function at every point on any surface (double integral is a special case of this) and add them all together.

Note that just like there is a scalar, and vector line integral, there is also a scalar and vector surface integral.

Before we define the scalar surface integral, we need to get better at describing surfaces in $3D$ space. Just like in line integrals, we want to parametrize a curve, here we want to parametrize a surface.

Definition 6.6.1 Parametric Surfaces

Given some surface S, there is a two-variable parameterization in the form:

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

You can interpret this as taking the $uv-$plane and mapping each point into some surface in $xyz-$space. The $uv-$plane is called the **parameter space** or **parameter domain** and is the set of points which can be plugged into $\vec{r}$.

Common parametrizations include:

- **Cylinder of Radius** R:

$$\vec{r}(u,v) = \begin{bmatrix} R\cos(u) \\ R\sin(u) \\ v \end{bmatrix}$$

- **Sphere of Radius** R:

$$\vec{r}(\phi,\theta) = \begin{bmatrix} R\cos(\theta)\sin(\phi) \\ R\sin(\theta)\sin(\phi) \\ R\cos(\phi) \end{bmatrix}$$

$$0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

- **Explicit Surface**

$$y = f(x,y)$$

$$\vec{r}(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$$

Surface Integral of a Scalar-Valued Function

We are now ready to define the scalar surface integral, and a method to evaluate it:

Definition 6.6.2 Scalar Surface Integral

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is:

$$\iint_S f(x,y,z)dS = \lim_{m,n\to\infty} \sum_{i=i}^{m} \sum_{j=1}^{n} f(\vec{r}(u^*,v^*))\Delta S_{ij}$$

Just like in line integrals, where we used:

$$ds = \left\| \vec{r}\,'(t) \right\| dt$$

... here we use an analogous equation:

$$dS = \left\| \vec{t}_u \times \vec{t}_v \right\| dA$$

Where:

$$\vec{t}_u = < \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} >$$
$$\vec{t}_v = < \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} >$$

... to create the following functional equation for the scalar surface integral:

$$\boxed{\iint_C f(x,y,z)ds = \iint_D f(\vec{r}(u,v)) \left\| \vec{t}_u \times \vec{t}_v \right\| dA}$$

... where D is the region of the parameter space which defines the surface.

Orientations of Surfaces

Just like curves can be oriented with line integrals, so can surfaces. At every point on a surface you can define two normal vectors $\vec{N}$ and $-\vec{N}$ which point in opposite directions. Choosing one gives an orientation to your curve. Given some orientable surface, we can define it's normal:

$$\vec{N} = \frac{\vec{t}_u \times \vec{t}_v}{\left\| \vec{t}_u \times \vec{t}_v \right\|}$$

A surface can be oriented if it has an "inner" and an "outer" side, or an "upward" and a "downward" side.

Surface Integral of a Vector Field

Imagine taking a surface in a vector field, chopping it up into small pieces, and calculating $\vec{F} \cdot \vec{N}$ at any point in that piece, doing that for all the pieces, and adding them all together. This idea is **flux through a surface**, and is what a vector surface integral calculates.

Definition 6.6.3 Vector Surface Integral

Let $\vec{F}$ be a continuous vector field with a domain that contains the oriented surface S with unit normal $\vec{N}$. Then the **surface integral** of $\vec{F}$ over S is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} dS$$

This type of integral is also called a **flux integral**.

A more functional form of this integral is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{t_u} \times \vec{t_v}) dA$$

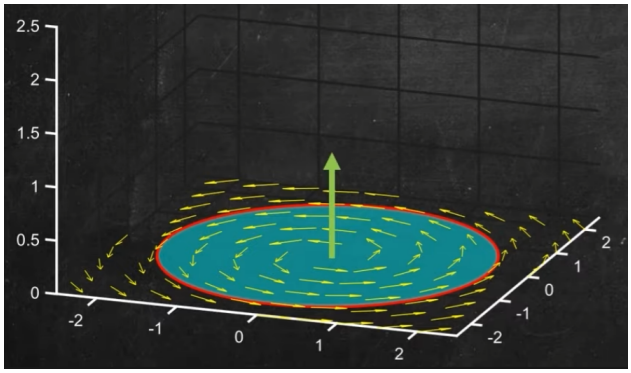
6.7 Stokes' Theorem

Stokes' theorem is a higher dimensional analogue to Green's theorem. Once again it allows us to relate an integral to its boundary conditions.

Recall that circulation form of Green's theorem can be stated as:

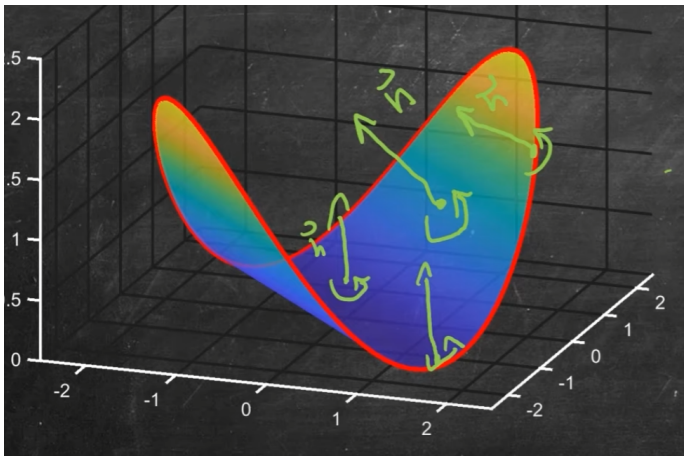
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot \hat{k} dA$$

This can be interpreted visually as:



As in the tendency of the field to curl around the point on the plane has an axis of rotation normal to the surface itself, and the double integral of this curling tendency across the region D is equal to the circulation integral around the boundary.

We now generalize this to curves which are not just on the $xy-$plane. Here, the normal is not always $\hat{k}$, as in:



If we want this to extend as expected, then the same logic should apply like the surface above. The tendency of the field to curl around a point **within the surface** has an axis of rotation $\vec{N}$ which is normal to the plane, and the double integral of this curling tendency across the **surface** S is equal to the circulation integral around the boundary.

It turns out, its true.

Theorem 6.7.1 Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation. If $\vec{F}$ is a vector field with continuous partial derivatives then on an open region containing S, then:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{N} dS$$

6.8 The Divergence Theorem

Continuing the trend we see in these theorems, that we can evaluate an integral by knowing information about the antiderivative at the boundary, we can imagine that the surface integral of a surface which encloses some volume is related in some way to the triple integral of the volume being enclosed. Recall the flux form of Green's theorem:

$$\oint_C \vec{F} \cdot \vec{N} ds = \iint_D \operatorname{div} \vec{F} dA$$

This tells us that given some region in the plane, we can determine the flux through its boundary by knowing the integral of the divergence inside. Imagine then taking the region and stretching it out into $xyz-$space. The same reasoning

holds in one higher dimension, we can know about the flux through the surface which was created by the boundary, by knowing information about the integral of the divergence in the **volume**.

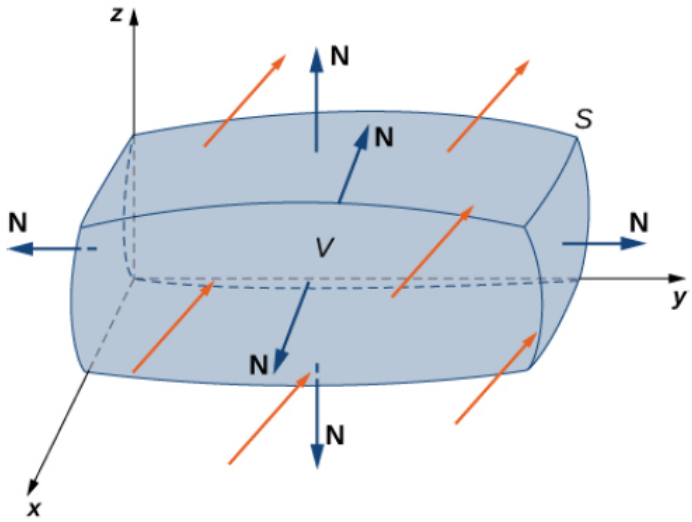


Figure 6.87 The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Theorem 6.8.1 The Divergence Theorem

Let S be a piecewise, smooth, closed surface that encloses a solid E. Assume that S is oriented outward, and let $\vec{F}$ be a vector field with continuous partial derivatives on an open region containing E. Then:

$$\iiint_E \operatorname{div} \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$$

Concluding Remarks

In summary, the following is a list of all the important theorems in Calculus we have learned so far:

- **The Fundamental Theorem of Calculus**:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

- **The Fundamental Theorem for Line Integrals:**

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$

- **Green's Theorem, circulation form:**

$$\iint_D (Q_x - P_y) dA = \int_C \vec{F} \cdot d\vec{r}$$

- **Green's Theorem, flux form:**

$$\iint_D (P_x + Q_y) dA = \int_C \vec{F} \cdot \vec{N} ds$$

- **Stokes' Theorem:**

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

- **The Divergence Theorem:**

$$\iiint_E \operatorname{div} \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$$

All applied in their particular domains, they are extensions of the Fundamental Theorem of Calculus into higher dimensions. The derivative, the gradient, the divergence, and the curl of their respective types of functions are all *differential operators*, as in they reason about the rate of change of the function in different ways. The unifying principle is:

`Integrating a differential operator acting on a field over a domain is equivalent to integrating the field components along the boundary.`

Conclusion

This concludes all the content in this course. I really hope these notes helped, I really enjoyed making them.

Good luck on the exam!

- Adam Szava